Impossibility Theorems in Graph Aggregation

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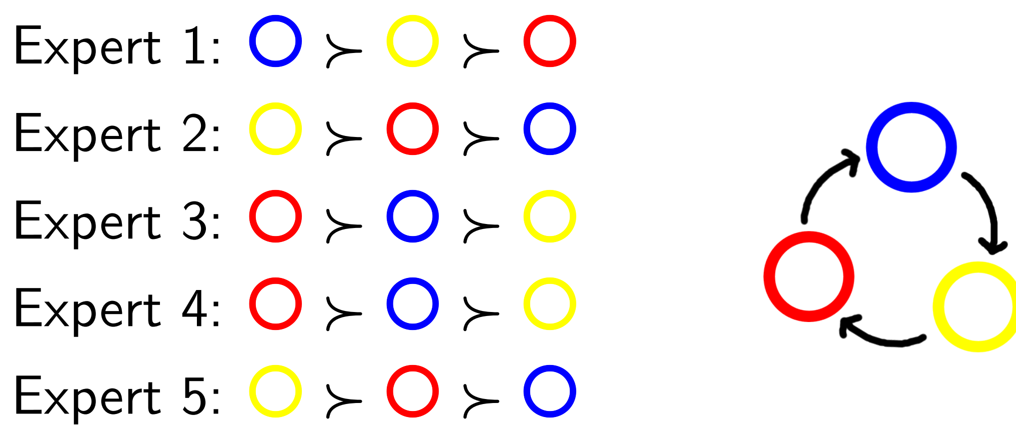
Social Choice and the Condorcet Paradox

Social Choice Theory asks: how should we aggregate the preferences of the members of a group to obtain a "social preference"?



Marie Jean Antoine Nicolas de Caritat (1743–1794), better known as the **Marquis de Condorcet**: Highly influential Mathematician, Philosopher, Political Scientist, Political Activist. Observed that the *majority rule* may produce inconsistent outcomes ("Condorcet Paradox").

Arrow's Impossibility Theorem

In 1951, K.J. Arrow published his famous *Impossibility Theorem:*

Any preference aggregation mechanism for *three* or more alternatives that satisfies the axioms of *Pareto* and *IIA* must be *dictatorial*.

- (Weak) Pareto: if everyone says $A \succ B$, then so should society.
- Independence of Irrelevant Alternatives (IIA): if society says $A \succ B$ and someone changes their ranking of C, then society should still say $A \succ B$.

Kenneth J. Arrow (born 1921): American Economist; Professor Emeritus of Economics at Stanford; Nobel Prize in Economics 1972 (youngest recipient ever). His 1951 PhD thesis started modern Social Choice Theory. Google Scholar lists 13,580 citations of the thesis.

Logic and Social Choice Theory

This talk will not be about logic. Just a few words:

Logic is relevant to social choice theory:

- Formal minimalism (Pauly, Synthese 2008)
- Verification of proofs (e.g., Nipkow, JAR 2009)
- Automation of tasks (Tang & Lin, AIJ 2009; Geist & E., JAIR 2011)

Much of classical social choice theory has been modelled in logic:

- Classical first-order logic (Grandi & E., JPL 2013)
- Tailor-made modal logics (e.g., Ågotnes et al., JAAMAS 2010)

But all of these approaches have some shortcomings:

- modelling of *universal domain* assumption not elegant
- set of *individuals* fixed to *specific size* (or at least not to any *finite* set)
- gap between logical modelling and suitability for *automated reasoning*

U. Endriss. Logic and Social Choice Theory. In A. Gupta and J. van Benthem (eds.), *Logic and Philosophy Today*, College Publications, 2011.

Talk Outline

- Graph Aggregation
- Collective Rationality
- A General Impossibility Result

Graph Aggregation

Fix a finite set of *vertices* V. A (directed) *graph* $G = \langle V, E \rangle$ based on V is defined by a set of *edges* $E \subseteq V \times V$.

Each member of a finite set of *individuals* $\mathcal{N} = \{1, \ldots, n\}$ provides such a graph, giving rise to a *profile* $\boldsymbol{E} = (E_1, \ldots, E_n)$.

An *aggregator* is a function mapping profiles to collective graphs:

$$F : (2^{V \times V})^n \to 2^{V \times V}$$

Example: *majority rule* (accept an edge *iff* $> \frac{n}{2}$ of the individuals do)

Axioms

We may want to impose certain *axioms* on $F : (2^{V\times V})^n \to 2^{V\times V}$, e.g.:

- *Anonymous:* $F(E_1, \ldots, E_n) = F(E_{\sigma(1)}, \ldots, E_{\sigma(n)})$
- *Nondictatorial:* for no $i^\star \in \mathcal{N}$ you always get $F(\boldsymbol{E}) = E_{i^\star}$
- *Unanimous:* $E \supseteq E_1 \cap \cdots \cap E_n$
- *Grounded:* $E \subseteq E_1 \cup \cdots \cup E_n$
- *Neutral:* $N_e^{\boldsymbol{E}} = N_{e'}^{\boldsymbol{E}}$ implies $e \in F(\boldsymbol{E}) \Leftrightarrow e' \in F(\boldsymbol{E})$
- *Independent:* $N_e^{\boldsymbol{E}} = N_e^{\boldsymbol{E'}}$ implies $e \in F(\boldsymbol{E}) \Leftrightarrow e \in F(\boldsymbol{E'})$

For technical reasons, we'll restrict some axioms to *nonreflexive edges* $(x, y) \in V \times V$ with $x \neq y$ (NR-neutral, NR-nondictatorial).

Notation: $N_e^{\boldsymbol{E}} = \{i \in \mathcal{N} \mid e \in E_i\}$ = *coalition* accepting edge e in $\boldsymbol{E}$

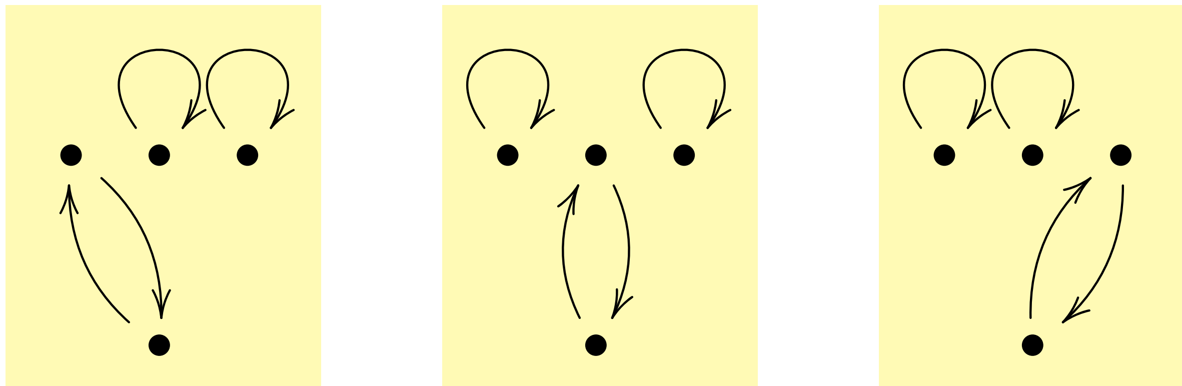
Collective Rationality

Aggregator F is *collectively rational* (CR) for graph property P if, whenever all individual graphs E_i satisfy P, so does the outcome $F(\boldsymbol{E})$.

Examples for *graph properties:* reflexivity, transitivity, seriality, ...

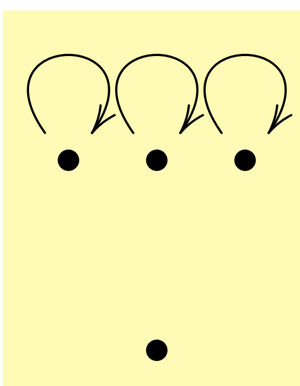
Example

Three agents each provide a graph on the same set of four vertices:



1 2 3

If we aggregate using the *majority rule*, we obtain this graph:



Observations:

- Majority rule not collectively rational for *seriality*.
- But *symmetry* is preserved.
- So is *reflexivity* (easy: individuals violate it).

A Simple Possibility Result

The fact that the example worked for reflexivity is no coincidence:

Proposition 1 *Any unanimous aggregator is CR for reflexivity.*

Proof: If every individual graph includes edge (x, x), then unanimity ensures the same for the collective outcome graph. ✓

Arrow's Theorem

Our formulation in graph aggregation:

> *For $|V| \geqslant 3$, there exists no NR-nondictatorial, unanimous, grounded, and independent aggregator that is CR for reflexivity, transitivity, and completeness.*

This implies the standard formulation, because:

- weak preference orders = reflexive, transitive, complete graphs
- nondictatorial = NR-nondictatorial for reflexive graphs
- unanimous + grounded $\Rightarrow$ (weak) Pareto
- CR for reflexivity is vacuous (implied by unanimity)

Main question for this talk:

- ▶ For what other classes of graphs does this go through?

Winning Coalitions

If an aggregator F is *independent*, then for every edge e there exists a set of *winning coalitions* $\mathcal{W}_e \subseteq 2^{\mathcal{N}}$ such that $e \in F(\boldsymbol{E}) \Leftrightarrow N_e^{\boldsymbol{E}} \in \mathcal{W}_e$.

Furthermore:

- If F is *unanimous*, then $\mathcal{N} \in \mathcal{W}_e$ for all edges e.
- If F is *grounded*, then $\emptyset \notin \mathcal{W}_e$ for all edges e.
- If F is *neutral*, then there is one $\mathcal{W}$ with $\mathcal{W} = \mathcal{W}_e$ for all edges e.

Proof Plan

Given: Arrovian aggregator F (*unanimous*, *grounded*, *independent*)
Want: Impossibility for *collective rationality* for graph property P

This will work if P is *contagious*, *implicative*, and *disjunctive* (TBD).

Lemma: CR for *contagious* $P \Rightarrow F$ is NR-*neutral*.

$\Rightarrow F$ characterised by some $\mathcal{W}$: $(x, y) \in F(\boldsymbol{E}) \Leftrightarrow N^{\boldsymbol{E}}_{(x,y)} \in \mathcal{W}$ $[x \neq y]$

Lemma: CR for *implicative* & *disjunctive* $P \Rightarrow \mathcal{W}$ is an *ultrafilter*, i.e.:

(i) $\emptyset \notin \mathcal{W}$ [this is immediate from groundedness]
(ii) $C_1, C_2 \in \mathcal{W}$ implies $C_1 \cap C_2 \in \mathcal{W}$ (closure under intersections)
(iii) C or $\mathcal{N} \setminus C$ is in $\mathcal{W}$ for all $C \subseteq \mathcal{N}$ (maximality)

$\mathcal{N}$ is *finite* $\Rightarrow \mathcal{W}$ is *principal*: $\exists i^\star \in \mathcal{N}$ s.t. $\mathcal{W} = \{C \in 2^{\mathcal{N}} \mid i^\star \in C\}$
But this just means that $i^\star$ is a dictator: F is (NR-)*dictatorial*. ✓

Neutrality Lemma

Consider any Arrovian aggregator (unanimous, grounded, independent).

Call a property P *xy/zw-contagious* if there exist sets $S^+, S^- \subseteq V \times V$ s.t. every graph $E \in P$ satisfies $[\bigwedge S^+ \wedge \neg \bigvee S^-] \to [xEy \to zEw]$.

CR for xy/zw-contagious P implies: coalition $C \in \mathcal{W}_{(x,y)} \Rightarrow C \in \mathcal{W}_{(z,w)}$

Call P *contagious* if it satisfies (at least) one of the three conditions below:

(i) P is xy/yz-contagious for all $x, y, z \in V$.
(ii) P is xy/zx-contagious for all $x, y, z \in V$.
(iii) P is xy/xz-contagious and xy/zy-contagious for all $x, y, z \in V$.

Example: *Transitivity* ($[yEz] \to [xEy \to xEz]$ and $[zEx] \to [xEy \to zEy]$)

Contagiousness allows us to reach every NR edge from every other NR edge. Thus, *CR for contagious P* implies $\mathcal{W}_e = \mathcal{W}_{e'}$ for all NR edges e, e'.

So: *Collective rationality* for a *contagious* property implies NR-*neutrality*.

Ultrafilter Lemma

Let F be *unanimous*, *grounded*, *independent*, NR-*neutral*, and *CR* for P.
So there exists a family of winning coalitions $\mathcal{W}$ s.t. $e \in F(\boldsymbol{E}) \Leftrightarrow N_e^{\boldsymbol{E}} \in \mathcal{W}$.

Show that $\mathcal{W}$ is an *ultrafilter* (under certain assumptions on P):

(ii) *Closure under intersections:* $C_1, C_2 \in \mathcal{W} \Rightarrow C_1 \cap C_2 \in \mathcal{W}$

Call P *implicative* if there exist $S^+, S^- \subseteq V \times V$ and $e_1, e_2, e_3 \in V \times V$ s.t. all graphs $E \in P$ satisfy $[\bigwedge S^+ \wedge \neg \bigvee S^-] \rightarrow [e_1 \wedge e_2 \rightarrow e_3]$.

Example: transitivity

CR for implicative $P \Rightarrow$ closure under intersections

Proof: consider profile where C_1 accept e_1, C_2 acc. e_2, $C_1 \cap C_2$ acc. e_3

(iii) *Maximality:* C or $\mathcal{N} \setminus C$ in $\mathcal{W}$ for all $C \subseteq \mathcal{N}$

Call P *disjunctive* if there exist $S^+, S^- \subseteq V \times V$ and $e_1, e_2 \in V \times V$ s.t. all graphs $E \in P$ satisfy $[\bigwedge S^+ \wedge \neg \bigvee S^-] \rightarrow [e_1 \vee e_2]$.

Example: completeness

CR for disjunctive $P \Rightarrow$ maximality

Proof: consider profile where C accept e_1, $\mathcal{N} \setminus C$ accept e_2

General Impossibility Theorem

We have sketched a proof for the following theorem:

Theorem 2 *For $|V| \geqslant 3$, there exists no NR-nondictatorial, unanimous, grounded, and independent aggregator that is CR for any graph property that is contagious, implicative, and disjunctive.*

Many combinations of properties have our meta-properties:

		C/I/D
Transitivity	$\forall xyz.(xEy \wedge yEz \rightarrow xEz)$	$++-$
Right Euclidean	$\forall xyz.(xEy \wedge xEz \rightarrow yEz)$	$++-$
Left Euclidean	$\forall xyz.(xEy \wedge zEy \rightarrow zEx)$	$++-$
Seriality	$\forall x.\exists y.xEy$	$--+$
Completeness	$\forall xy.[x \neq y \rightarrow (xEy \vee yEx)]$	$--+$
Connectedness	$\forall xyz.[xEy \wedge xEz \rightarrow (yEz \vee zEy)]$	$+++$
Negative Transitivity	$\forall xyz.[xEy \rightarrow (xEz \vee zEy)]$	$+-+$

Last Slide

We have introduced *graph aggregation* as a generalisation of preference aggregation and then considered *collective rationality*.

Why is this interesting?

- Potential for *applications:* abstract argumentation, social networks
- Deep insights into the *structure of impossibilities:* direct link between CR requirements and neutrality/ultrafilter conditions