Complexity Results on Dependence Logic

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Outline of the talk

1. Short introduction to dependence logic
2. Some variants of the dependence atoms
3. The Strict and the Lax semantics
4. Recent results on expressive power, satisfiability and model checking
5. Conclusion

Dependence logic

Definition

The syntax of dependence logic ($\mathcal{D}$) extends the syntax of FO, defined in terms of $\vee$, $\wedge$, $\neg$, $\exists$ and $\forall$, by new atomic (dependence) formulas of the form

$$=\!(t_1, \ldots, t_n), \tag{1}$$

where $t_1, \ldots, t_n$ are terms.

In (1), n is called the arity (or width) of the dependence atom.

Semantics of $\mathcal{D}$

The semantics of $\mathcal{D}$ is defined in terms of *teams* (sets of assignments):

Definition

Let A be a set and $\{x_1, \ldots, x_k\}$ variables. A *team* X of A with domain $\{x_1, \ldots, x_k\}$ is a set of assignments s,

$$s\colon \{x_1, \ldots, x_k\} \to A.$$

Semantics of $\mathcal{D}$

The following operations are used to interpret quantifiers in $\mathcal{D}$. Below, $s(a/x_n)$ is the assignment that agrees otherwise with s, but maps x_n to a.

Definition

Suppose A is a set, X is a team of A, and $F\colon X \to A$.

- *supplementation*: $X(F/x) = \{s(F(s)/x) \ : s \in X\}$.
- *Duplication*: $X(A/x) = \{s(a/x) \ : s \in X \ \text{ and } a \in A\}$.

Supplementation

Let $A = \{0, 1\}$ and X be

	x_0	x_1
s_0	1	0
s_1	0	1

Let $F: X \to A$ be such that $F(s_0) = 1$ and $F(s_1) = 0$, then $X(F/x_2)$ is

	x_0	x_1	x_2
s_2	1	0	1
s_3	0	1	0

Duplication

Let $A = \{0, 1\}$ and X be

	x_0	x_1
s_0	1	0
s_1	0	1

Then $X(A/x_2)$ is

	x_0	x_1	x_2
s_2	1	0	1
s_3	1	0	0
s_4	0	1	1
s_5	0	1	0

Satisfaction for NNF-formulas

Definition

Below $\phi(t_1, \ldots, t_n)$ is atomic or negated atomic FO-formula:

- $\mathfrak{A} \models_X \phi(t_1, \ldots, t_n) \Leftrightarrow$ for all $s \in X$: $\mathfrak{A} \models_s \phi(t_1, \ldots, t_n)$
- $\mathfrak{A} \models_X =\!(t_1, \ldots, t_n) \Leftrightarrow$ for all $s, s' \in X$: if $t_i^{\mathfrak{A}}\langle s \rangle = t_i^{\mathfrak{A}}\langle s' \rangle$ for $1 \leq i \leq n-1$, then $t_n^{\mathfrak{A}}\langle s \rangle = t_n^{\mathfrak{A}}\langle s' \rangle$.
- $\mathfrak{A} \models_X \psi \wedge \phi \Leftrightarrow \mathfrak{A} \models_X \psi$ and $\mathfrak{A} \models_X \phi$.
- $\mathfrak{A} \models_X \psi \vee \phi \Leftrightarrow X = Y \cup Z$ such that $\mathfrak{A} \models_Y \psi$ and $\mathfrak{A} \models_Z \phi$
- $\mathfrak{A} \models_X \exists x \psi \Leftrightarrow \mathfrak{A} \models_{X(F/x)} \psi$ for some $F\colon X \to A$.
- $\mathfrak{A} \models_X \forall x \psi \Leftrightarrow \mathfrak{A} \models_{X(A/x)} \psi$.

Finally, a sentence φ of $\mathcal{D}$ is true in $\mathfrak{A}$ if $\mathfrak{A} \models_{\{\emptyset\}} \varphi$.

Example

Not all familiar propositional equivalences of connectives hold for $\mathcal{D}$, e.g., idempotence of disjunction, and the distributivity laws of disjunction and conjunction fail.

Example

Let $A = \{0, 1, 2\}$ and X be

	x_0	x_1	x_2
s_0	1	2	2
s_1	2	1	2
s_2	2	0	2

(2)

Now $\mathfrak{A} \not\models_X x_0 = x_2$ and $\mathfrak{A} \not\models_X \neg x_0 = x_2$. Also $\mathfrak{A} \not\models_X =\!(x_2, x_0)$, but $\mathfrak{A} \models_X (=\!(x_2, x_0) \vee =\!(x_2, x_0))$.

Basic properties of $\mathcal{D}$

Proposition

Let ϕ be a formula of $\mathcal{D}$ without dependence atoms. Then for all $\mathfrak{A}$ and X:

- $\mathfrak{A} \models_X \phi \Leftrightarrow$ *for all* $s \in X$: $\mathfrak{A} \models_s \phi$

Proposition (Downward closure)

Let $Y \subseteq X$ teams. Then $\mathfrak{A} \models_X \phi$ implies $\mathfrak{A} \models_Y \phi$.

Proposition (Locality)

Let $\mathit{Fr}(\phi) \subseteq V$. Then, for all models $\mathfrak{A}$ and teams X, $\mathfrak{A} \models_X \phi$ if and only if $\mathfrak{A} \models_{X \upharpoonright V} \phi$.

Definition (Grädel and Väänänen, 2013)

Conditional independence atom

$$\overline{y} \perp_{\overline{x}} \overline{z}$$

interpreted as: $\mathfrak{A} \models_X \overline{y} \perp_{\overline{x}} \overline{z}$ iff for all $s, s' \in X$ s.t. $s(\overline{x}) = s'(\overline{x})$ there exists $s'' \in X$ s.t. $s''(\overline{xy}) = s(\overline{xy})$, and $s''(\overline{xz}) = s'(\overline{xz})$.

The version $\overline{y} \perp \overline{z}$, where $\overline{x} = \emptyset$, is called *Pure*.
$\mathrm{FO}(\perp_c)$ ($\mathrm{FO}(\perp)$) is FO with (pure) independence atoms.

Definition (Galliani 2012)

1. Inclusion atom

$$\overline{x} \subseteq \overline{y}$$

 interpreted as: $\mathfrak{A} \models_X \overline{x} \subseteq \overline{y}$ iff for all $s \in X$ there exists $s' \in X$ s.t. $s(\overline{x}) = s'(\overline{y})$.

2. Exclusion atom

$$\overline{x}|\overline{y}$$

 interpreted as: $\mathfrak{A} \models_X \overline{x}|\overline{y}$ iff $s(\overline{x}) \neq s'(\overline{y})$ for all $s, s' \in X$

The variants of $\mathcal{D}$ with the atoms $\overline{x} \subseteq \overline{y}$ and $\overline{x}|\overline{y}$, are denoted by $\mathrm{FO}(\subseteq)$, $\mathrm{FO}(|)$, and $\mathrm{FO}(\subseteq, |)$.

Strict and Lax semantics

Galliani 2012

The *Strict semantics* is obtained by changing the clause for $\vee$ to:

- $\mathfrak{A} \models_X \psi \vee \theta$ iff there are Y and Z such that $Y \cup Z = X$, $Y \cap Z = \emptyset$, and $\mathfrak{A} \models_Y \psi$ and $\mathfrak{A} \models_Z \theta$

The *Lax semantics* is obtained by changing the clause for $\exists$ to:

- $\mathfrak{A} \models_X \exists x\psi$ iff there exists $F : X \to \mathcal{P}(M)\setminus\{\emptyset\}$ such that $\mathfrak{A} \models_{X(F/x)} \psi$.

Downward closure renders the strict and the lax semantics equivalent for all $\mathcal{D}$-formulas.

Comparing the strict and the lax semantics I

The Locality property holds for all of the the aforementioned logics under the lax semantics:

Proposition (Locality for the lax semantics)

Let ϕ be a formula of $\mathrm{FO}(=(\ldots), \perp_{\mathrm{c}}, \subseteq)$ whose free variables $Fr(\phi)$ are contained in V. Then, for all models $\mathfrak{A}$ and teams X, $\mathfrak{A} \models_X \phi$ if and only if $\mathfrak{A} \models_{X \upharpoonright V} \phi$.

Comparing the strict and the lax semantics II

Under the strict semantics,

$$X \models x \subseteq y \vee z \subseteq y$$

but

$$X \upharpoonright \{x, y, z\} \not\models x \subseteq y \vee z \subseteq y,$$

where X is

	x	y	z	v
s_0	0	1	2	3
s_1	1	0	1	3
s_2	1	0	1	4
s_3	2	1	0	4

Comparing the strict and the lax semantics III

Note that $X \upharpoonright \{x, y, z\}$ is the team

	x	y	z
s_0	0	1	2
s_1	1	0	1
s_3	2	1	0

The claim follows because the full team $X \upharpoonright \{x, y, z\}$, and none of its singletons, satisfy $x \subseteq y$ or $z \subseteq y$.

Relations among logics

For both versions of the semantics, compositional translations of formulas show:

- $\mathcal{D} = \mathrm{FO}(|)$
- $\mathrm{FO}(\perp_c) = \mathrm{FO}(\perp) = \mathrm{FO}(\subseteq, |)$

Theorem (Galliani and Hella 2013)

Under the lax semantics, $\mathrm{FO}(\subseteq) = \mathrm{GFP}^+$.

Hence, over (ordered) finite structures, $\mathrm{FO}(\subseteq) = \mathrm{LFP} = \mathrm{PTIME}$. On the other hand, for the strict semantics:

Theorem (Hannula, K., Galliani 2013)

Under the strict semantics, $\mathrm{FO}(\subseteq) = \mathrm{ESO}$.

Complexity of syntactic fragments

Theorem (Jarmo Kontinen 2013)

Define

- *Let* φ *be* $=\!(x,y) \vee =\!(z,u)$
- *Let* ψ *be* $=\!(x,y) \vee =\!(z,u) \vee =\!(z,u)$

Deciding whether a finite team X *satisfies* φ *is NL-complete and, for* ψ*, NP-complete.*

By the above the universal $\mathcal{D}$-sentence of vocabulary $\{R\}$:

$$\forall xyzy(\neg R(x,y,z,u) \vee \psi),$$

defines an NP-complete problem. On the other hand, purely existential sentences of $\mathcal{D}$ are equivalent to FO-sentences.

Relevant fragments of ESO I

- Let $\mathrm{ESO}_f(k\text{-ary})$ denote the class of ESO-sentences

$$\exists f_1 \ldots \exists f_n \psi$$

in which the function symbols f_i are at most k-ary and ψ is first-order.

- Let $\mathrm{ESO}_f(k\forall)$ denote the class of skolem normal form ESO-sentences (i.e., ψ is quantifier free)

$$\exists f_1 \ldots \exists f_n \forall x_1 \ldots \forall x_m \psi,$$

where $m \leq k$.

Relevant fragments of ESO II

Theorem (Ajtai 1983)

Let R be a $k+1$-ary relation symbol. Then the property "$|R|$ even" cannot be defined in the logic $\mathrm{ESO}_f(k\text{-ary})$ but is definable in $\mathrm{ESO}_f(k+1\text{-ary})$.

Theorem (Grandjean and Olive 2004)

$\mathrm{ESO}_f(k\forall) = \mathrm{NTIME}_{\mathrm{RAM}}(n^k)$

Note that $\mathrm{NTIME}_{\mathrm{RAM}}(n^k) < \mathrm{NTIME}_{\mathrm{RAM}}(n^{k+1})$.

Fragments in Logics with Teams Semantics

Definition

Let $\mathcal{C} \subseteq \{\subseteq, =(\ldots), \perp_{\mathrm{c}}, \perp\}$.

- $\mathrm{FO}(\mathcal{C})(k\forall)$ is the class of $\mathrm{FO}(\mathcal{C})$ formulae in which at most k universal quantifiers may appear,
- $\mathrm{FO}(\mathcal{C})(k\text{-inc})$ is the class of $\mathrm{FO}(\mathcal{C})$ formulae in which inclusion atoms of the form $\vec{x}_1 \subseteq \vec{x}_2$ where $\vec{x}_1$ and $\vec{x}_2$ are sequences of length at most k, may appear,
- $\mathrm{FO}(\mathcal{C})(k\text{-dep})$ is the class of $\mathrm{FO}(\mathcal{C})$ formulae in which dependence atoms of the form $=(\vec{x}_1, x_2)$ where $\vec{x}_1 x_2$ is a sequence of length at most $k+1$, may appear,
- $\mathrm{FO}(\mathcal{C})(k\text{-ind})$ is the class of $\mathrm{FO}(\mathcal{C})$ formulae in which conditional independence atoms of the form $\vec{x}_2 \perp_{\vec{x}_1} \vec{x}_3$ where $\vec{x}_1\vec{x}_2\vec{x}_3$ is a sequence listing at most $k+1$ distinct variables, may appear.

The expressive power of the fragments I

Under the Lax semantics

By restricting the arity we get:

Theorem (Galliani, Hannula, and K. 2013; Durand and K. 2012)

$\mathrm{FO}(\perp_c)(k\text{-ind}) = \mathrm{ESO}_f(k\text{-ary}) = \mathcal{D}(k\text{-dep})$

On the other hand, for inclusion logic the following holds:

Theorem (Hannula 2014)

Over graphs,

$$\mathrm{FO}(\subseteq)(k\text{-ind}) < \mathrm{FO}(\subseteq)(k+1\text{-ind})$$

The expressive power of the fragments II
Under the Lax semantics

By restricting the number of universal quantifiers:

Theorem (Durand and K. 2012)

$$\mathrm{ESO}_f(k\forall) \leq \mathcal{D}(2k\forall) \leq \mathrm{ESO}_f(2k\forall).$$

In the presence of inclusion or independence atoms, universal quantifiers can be simulated by existential quantifiers:

Theorem (Hannula 2014; Galliani, Hannula and K. 2013)

1. *If* $\subseteq \in \mathcal{C}$ *then the hierarchy collapses at level* 1*:* $\mathrm{FO}(\mathcal{C}) = \mathrm{FO}(\mathcal{C})(1\forall)$*;*
2. *If* $\perp \in \mathcal{C}$ *then it collapses at level 2:* $\mathrm{FO}(\mathcal{C}) = \mathrm{FO}(\mathcal{C})(2\forall)$.

Hierarchy theorems for the lax semantics

	Arity of atoms	Number of $\forall$
$\mathcal{D}$	Strict	Infinite
$\mathrm{FO}(\perp_{\mathrm{c}})$	strict	collapse at $k = 2$
$\mathrm{FO}(\subseteq)$	strict	collapse at $k = 1$

The expressive power of the fragments
Under the strict semantics

The situation is more complicated because of the locality property failing.

Theorem (Hannula and K. 2014)

1. $\mathrm{FO}(\subseteq)(k\forall) = \mathrm{ESO}_f(k\forall) = \mathrm{NTIME}_{\mathrm{RAM}}(n^k)$
2. $\mathrm{FO}(\perp_c)(k\forall) \leq \mathrm{ESO}_f(k+1\forall)$
3. $\mathrm{ESO}_f(k\forall) \leq \mathrm{FO}(\perp_c)(2k\forall)$

This implies the following hierarchy theorems:

	Arity of atoms	Number of $\forall$
$\mathrm{FO}(\perp_c)$	?	infinite
$\mathrm{FO}(\subseteq)$	?	strict

The 2-variable fragment of $\mathcal{D}$

Denote by $\mathcal{D}^2$ the sentences of $\mathcal{D}$ in which only variables x and y appear.

Theorem (K., Kuusisto, Lohmann and Virtema 2011)

1. *The Satisfiability (and Finite Satisfiability) problem of $\mathcal{D}^2$ is NEXPTIME-complete.*
2. *The logic $\mathcal{D}^2$ is quite expressive being able to express, e.g., "$\mathfrak{A}$ infinite", "$|P| = |Q|$", and some NP-complete problems.*
3. *In contrast, the satisfiability (and finite satisfiability) problem of IF^2 is undecidable.*

Remark

The complexity of the validity problem for $\mathcal{D}^2$ is open.

Complexity of Model Checking

E. Grädel (2013) formulated a general model checking game for logics with team semantics.
Recall that the model checking problem, with input $(\phi, \mathfrak{A}, X)$, is to decide whether $\mathfrak{A} \models_X \phi$.

Theorem (Grädel 2013)

The model-checking problem for $\mathcal{D}$ is NEXPTIME-complete.

Futhermore, containment in NEXPTIME was shown to hold for any variant $\mathrm{FO}(\mathcal{C})$ of $\mathcal{D}$ s.t. the atoms in $\mathcal{C}$ are PTIME-computable.

Other complexity results

- A certain *Horn* fragment of $\mathcal{D}$ captures PTIME over successor structures (Ebbing, K. Müller and Vollmer 2012).
- *Intuitionistic implication* $\rightarrow$ makes $\mathcal{D}$ equivalent to full second-oder logic (Yang 2013). A similar result holds when dependence logic is extended by the classical negation.