Academy Colloquium: Dependence Logic,

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The Expressive Power of Unany Inclusion and Exclusion Atoms

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I denote inclusion logic containing only unary atoms by INC[1]. Similarly I use EXC[1] and INEX[1] for exclusion logic and inclusion-exclusion logic containing only unary atoms.

Nondependence atoms

n-ary nondependence atoms $\neq(t_1, \ldots, t_n)$
have the following truth condition:

$$\mathcal{M} \models_X \neq(t_1, \ldots, t_n), \text{ iff for all } s \in X \text{ there exists an } s' \in X, \text{ s.t.} \\ (s(t_1), \ldots, s(t_{n-1})) = (s'(t_1), \ldots, s'(t_{n-1})), \\ \text{but } s(t_n) \neq s'(t_n).$$

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Unary nondependence atoms are also called *inconstancy atoms*.

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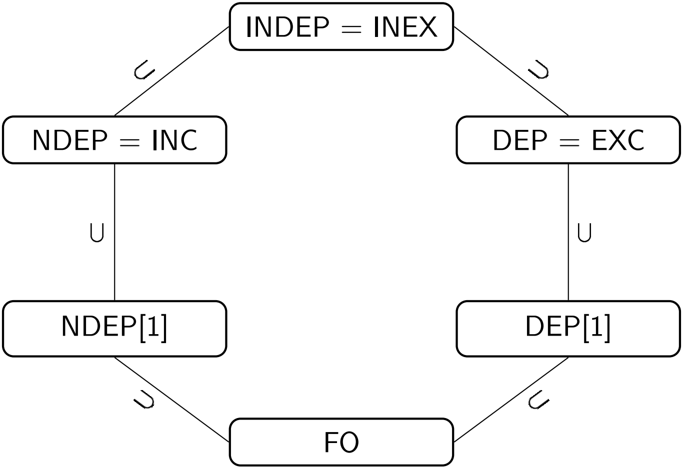
Fact 2 (Galliani) On the level of sentences:
DEP[1] = FO = NDEP[1].

The expressive power of logics on the level of formulas



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$\mathcal{M} \models_X t_1 \mid t_2$, iff

$$\mathcal{M} \models_X \forall y\, \exists w_1\, \exists w_2 \big(=\!(w_1) \wedge =\!(y, w_2) \wedge ((w_1 = w_2 \wedge y \neq t_1) \vee (w_1 \neq w_2 \wedge y \neq t_2)) \big).$$

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Corollary DEP[1] $\subseteq$ EXC[1] $\subseteq$ DEP[2].

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Existential quantification over the complement of the values of a given set of terms:

$$\mathcal{M} \vDash_X (\exists x \mid \bigcup_{i=1}^{n} t_i)\, \varphi,$$

$$\text{iff there exists } f : X \to \overline{\bigcup_{i=1}^{n} X(t_i)}, \text{ s.t. } \mathcal{M} \vDash_{X[f/x]} \varphi.$$

Some properties of graphs expressible in EXC[1]

Undirected graph $\mathcal{G} = (V, E)$ is disconnected, iff

$$\mathcal{G} \models \forall\, x \,\exists\, y_1 (\exists\, y_2 \mid y_1)\big((x = y_1 \vee x = y_2) \\ \wedge (\forall\, z_1 \subseteq y_1)(\forall\, z_2 \subseteq y_2) \neg E z_1 z_2\big).$$

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Corollary EXC[1] $\nsubseteq$ DEP[1], and thus DEP[1] $\subset$ EXC[1].

Expressing EXC[1] with EMSO

Let φ be EXC[1]-sentence. We label all the instances of exclusion atoms $(t_1 \mid t_2)_1, \ldots, (t_1 \mid t_2)_n$ occuring in φ. Now it holds:

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where φ' is defined recursively:

$$\begin{aligned}
(\psi)' &= \psi, \text{ if } \psi \text{ is a literal} \\
((t_1 \mid t_2)_i)' &= P_i t_1 \wedge \neg P_i t_2, \text{ for all } i \in \{1, \ldots, n\} \\
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Corollary EXC[1] $\subseteq$ EMSO.

Comparing EMSO and DEP[2]

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So we have shown that: DEP[1] $\subset$ EXC[1] $\subset$ DEP[2].

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$\mathcal{M} \models_X t_1 \subseteq t_2$, iff

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where φ' and φ'_i ($i \in \{1, \dots, n\}$) are defined recursively:

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(\psi)' &= \psi, \text{ if } \psi \text{ is a literal} \\
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Corollary INC[1] $\subseteq$ EMSO.

A NDEP[2]-formula which is not expressible in INC[1]

Let $L = \emptyset$ and $\mathcal{M}$ be an L-model, s.t. $\mathrm{dom}(\mathcal{M}) = \{0, 1\}$.
Let $X = \{s_1, s_2\}$ and $Y = \{s_1, s_2, s_3\}$, where

$$s_1 = \{(x, 0), (y, 0)\}$$
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By using the Ehrenfeucht-Fraïssé game for inclusion logic (presented by Galliani and Hella) we can show that there exists no INC[1]-formula φ so that $\mathcal{M} \models_X \varphi$, but $\mathcal{M} \not\models_Y \varphi$.

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Let $X = \{s_1, s_2\}$ and $Y = \{s_1, s_2, s_3\}$, where

$$s_1 = \{(x, 0), (y, 0)\}$$
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Clearly now $\mathcal{M} \models_X {\neq}(x, y)$, but $\mathcal{M} \not\models_Y {\neq}(x, y)$.

By using the Ehrenfeucht-Fraïssé game for inclusion logic (presented by Galliani and Hella) we can show that there exists no INC[1]-formula φ so that $\mathcal{M} \models_X \varphi$, but $\mathcal{M} \not\models_Y \varphi$.

Corollary NDEP[2] $\not\subseteq$ INC[1], and thus INC[1] $\subset$ NDEP[2].

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Corollary NDEP[2] $\nsubseteq$ INC[1], and thus INC[1] $\subset$ NDEP[2].

So we have shown that: NDEP[1] $\subset$ INC[1] $\subset$ NDEP[2].

Expressing INEX[1] with EMSO

Let φ be INEX[1]-sentence. We label all the instances of exclusion atoms $(t_1 \mid t_2)_1, \ldots, (t_1 \mid t_2)_n$ occuring in φ, and define φ':

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\begin{aligned}
(\psi)' &= \psi, \text{ if } \psi \text{ is a literal} \\
((t_1 \mid t_2)_i)' &= P_i t_1 \wedge \neg P_i t_2, \text{ for all } i \in \{1, \ldots, n\} \\
(t_1 \subseteq t_2)' &= t_1 \subseteq t_2 \\
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Because φ' is an INC[1]-sentence, it is equivalent with some EMSO-sentence μ. Now it holds: $\mathcal{M} \models \varphi$, iff $\mathcal{M} \models \exists P_1 \ldots \exists P_n\, \mu$.

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Corollary INEX[1] $\subseteq$ EMSO.

Term value preserving disjunction

The following operator can be expressed in INEX[1]-logic:

$\mathcal{M} \models_X \varphi \vee_{t_1 \ldots t_n} \psi$, iff there exists $Y, Y' \subseteq X$ s.t.

$$Y \cup Y' = X,\ \mathcal{M} \models_Y \varphi \text{ and } \mathcal{M} \models_{Y'} \psi,$$
$$\text{and if } Y, Y' \neq \emptyset, \text{ then } Y(t_i) = Y'(t_i) = X(t_i),$$
$$\text{for all } i \in \{1, \ldots, n\}.$$

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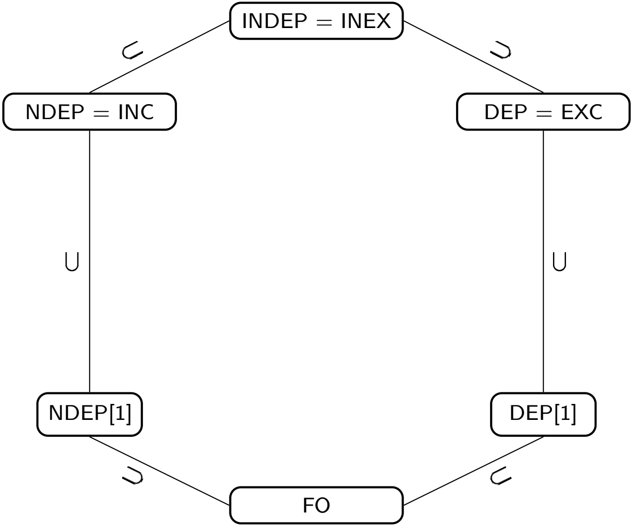
Corollary EMSO $\subseteq$ INEX[1], and thus INEX[1] $=$ EMSO.

The expressive power of logics on the level of formulas



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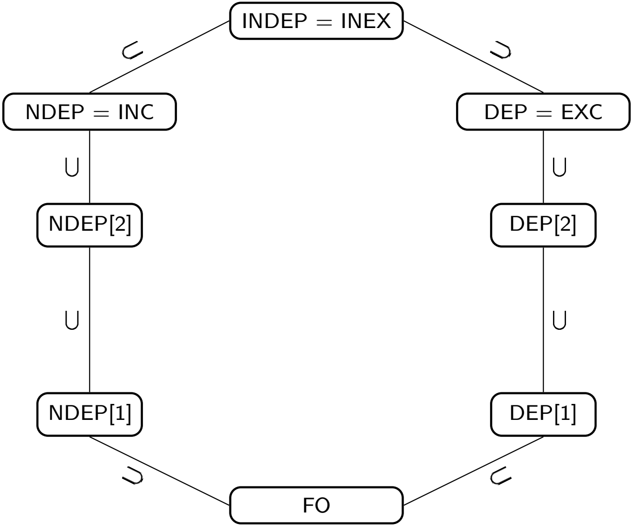

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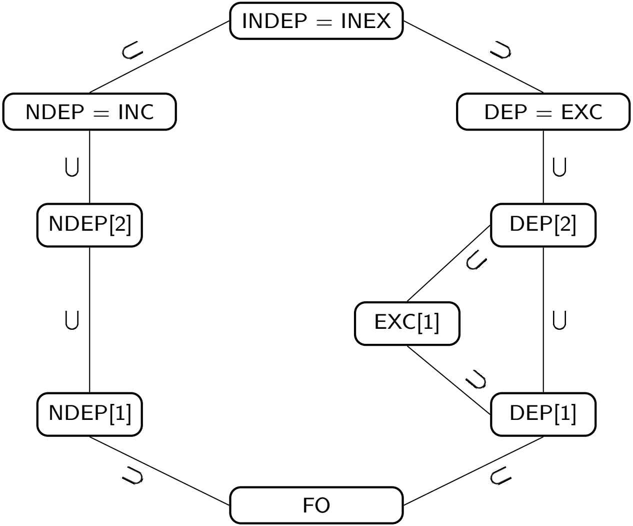

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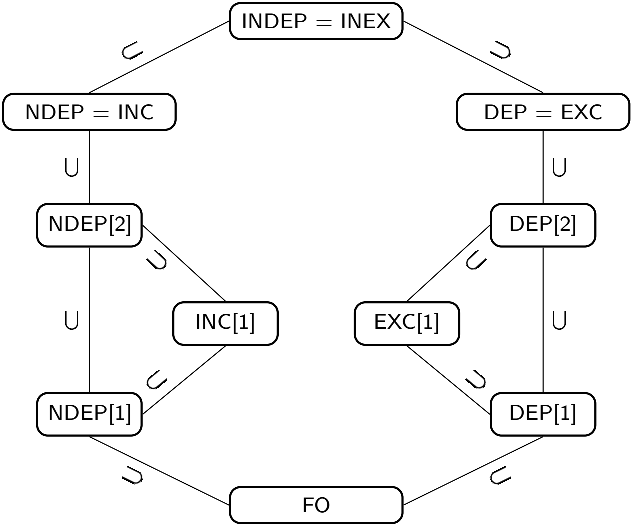

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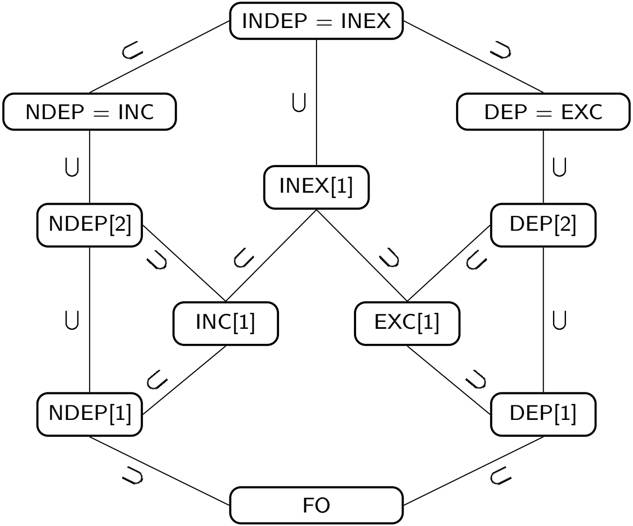

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Appendix: Expressing some operators

The intuitionistic disjunction can be defined in EXC[1]:

$$\begin{aligned}\varphi \sqcup \psi := \big(\forall z_1 \forall z_2 \,(z_1 = z_2) \wedge (\varphi \vee \psi)\big) \\ \vee \exists z_1 \exists z_2 \,\big({=}(z_1) \wedge {=}(z_2) \\ \wedge ((z_1 = z_2 \wedge \varphi) \vee (z_1 \neq z_2 \wedge \psi))\big)\end{aligned}$$

It has the following truth condition:

$$\mathcal{M} \models_X \varphi \sqcup \psi, \text{ iff } \mathcal{M} \models_X \varphi \text{ or } \mathcal{M} \models_X \psi.$$

Appendix: Expressing some operators

Following operators can be defined in EXC[1]:

$$(\exists\, x \mid \bigcup_{i=1}^{n} t_i)\, \varphi := \exists\, x \left(\bigwedge_{i=1}^{n} x \mid t_i \wedge \varphi \right)$$

$$(\forall\, x \subseteq t)\, \varphi := \forall\, x\, \varphi \sqcup \forall\, x\, (\exists\, y \mid t)(x = y \vee \varphi)$$

Note: The latter operator cannot be defined similarly in INEX[1], because it requires downward closure to work as expected.

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The following operator can be defined in INEX[1]:

$$
\begin{aligned}
\varphi \vee_{t_1 \ldots t_n} \psi :=& \\
&(\varphi \sqcup \psi) \sqcup \exists\, y_1 \exists\, y_2 \big(=\!(y_1) \wedge =\!(y_2) \wedge y_1 \neq y_2 \\
&\qquad \wedge \exists\, x \big(\bigwedge_{i=1}^{n} (\theta_i \sqcup \theta'_i) \wedge \big((x = y_1 \wedge \varphi) \vee (x = y_2 \wedge \psi) \big) \big) \big),
\end{aligned}
$$

where θ_i and θ'_i ($i \in \{1, \ldots, n\}$) are:

$$
\begin{aligned}
\theta_i &:= (x = y_1 \wedge \forall\, z\, (z \subseteq t_i)) \vee (x = y_2 \wedge \forall\, z\, (z \subseteq t_i)) \\
\theta'_i &:= (\exists\, u \mid t_i)\, \exists\, w_1 \exists\, w_2 \big(\big((x = y_1 \wedge w_1 = t_i \wedge w_2 = u) \\
&\qquad \vee (x = y_2 \wedge w_1 = u \wedge w_2 = t_i) \big) \wedge t_i \subseteq w_1 \wedge t_i \subseteq w_2 \big)
\end{aligned}
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