Axiomatizing propositional dependence logic

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Outline

1. Propositional dependence logic
2. Axiomatizing $\mathbf{PD}^{\vee}$
3. Axiomatizing $\mathbf{PD}$

Propositional Dependence Logic

- Syntax of first-order dependence logic

$$\phi ::= \alpha \mid {=}(t_0, \ldots, t_n) \mid \phi \wedge \phi \mid \phi \otimes \phi \mid \forall x \phi \mid \exists x \phi$$

  where α is a first-order literal and $t_0, \ldots, t_n$ are first-order terms.

- Syntax of propositional dependence logic (**PD**):

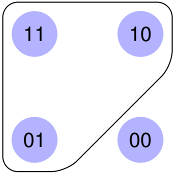
$$\phi ::= p_i \mid \neg p_i \mid {=}(p_{i_0}, \ldots, p_{i_k}) \mid \phi \wedge \phi \mid \phi \otimes \phi$$

  where $p_i, p_{i_0}, \ldots, p_{i_k}$ are propositional variables.

- $\mathbf{PD}^{\vee}$ is the logic extended from **PD** by adding the intuitionistic disjunction $\vee$.

Team

- A *team* is a set of valuations, i.e. a set of functions $s : \mathbb{N} \to \{0, 1\}$.
- For any n-element subset N of $\mathbb{N}$, a function $s : N \to \{0, 1\}$ is called an *n-valuation* on N.
- An *n-team* on N is a set of n-valuations on N.



Fixing $N = \{i_1, \dots, i_n\}$, there are in total 2^n distinct n-valuations, and 2^{2^n} n-teams, among which there exists a biggest team (denoted by $\mathbf{2^n}$) consisting of all n-valuations on N.

A formula of the form $\phi(p_{i_1}, \dots, p_{i_n})$ is called an *n-formula*. The truth of $\phi(p_{i_1}, \dots, p_{i_n})$ on a team X depends only on the valuations of X on $\{i_1, \dots, i_n\}$.

Team Semantics

Let X be a team.

- $X \models p_i$ iff $s(i) = 1$ for all $s \in X$
- $X \models \neg p_i$ iff $s(i) = 0$ for all $s \in X$

- $X \models =\!(p_{i_0}, \ldots, p_{i_{k-1}}, p_{i_k})$ iff for any $s_1, s_2 \in X$,

$$\left[\, s_1(i_0) = s_2(i_0), \ldots, s_1(i_{k-1}) = s_2(i_{k-1}) \,\right] \Longrightarrow s_1(i_k) = s_2(i_k)$$

- $X \models \varphi \wedge \psi$ iff $X \models \varphi$ and $X \models \psi$

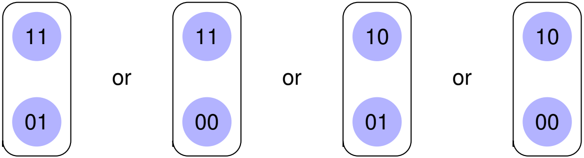
- $X \models \phi \otimes \psi$ iff there exist $Y, Z \subseteq X$ s.t. $X = Y \cup Z$,

  $Y \models \phi$ and $Z \models \psi$.

- $X \models \phi \vee \psi$ iff $X \models \phi$ or $X \models \psi$

Dependence atoms are definable using $\vee$

$$=\!(p_0, p_1) \equiv \Big((p_0 \wedge p_1) \otimes (\neg p_0 \wedge p_1)\Big) \vee \Big((p_0 \wedge p_1) \otimes (\neg p_0 \wedge \neg p_1)\Big)$$
$$\vee \Big((p_0 \wedge \neg p_1) \otimes (\neg p_0 \wedge p_1)\Big) \vee \Big((p_0 \wedge \neg p_1) \otimes (\neg p_0 \wedge \neg p_1)\Big)$$



In general, we have that:

$$=\!(p_{j_0}, \dots, p_{j_{k-1}}, p_{j_k}) \equiv \bigvee_{f \in 2^{\mathbf{2^k}}} \bigotimes_{s \in \mathbf{2^k}} \left(p_{j_0}^{s(j_0)} \wedge \dots \wedge \dots p_{j_{k-1}}^{s(j_{k-1})} \wedge p_{j_k}^{f(s)}\right)$$

where $p^1 := p$, $p^0 := \neg p$ and $\mathbf{2^k}$ is the maximal k-team on $\{j_0, \dots, j_{k-1}\}$.

The logics **PD** and $\mathbf{PD}^{\vee}$

- are downwards closed, that is,
$$X \models \phi \text{ and } Y \subseteq X \Longrightarrow Y \models \phi;$$
- and have the empty team property, that is, $\emptyset \models \phi$ for all ϕ.

It follows that fixing $N = \{i_1, \ldots, i_n\}$, the *semantic truth set*
$$[\![\phi]\!] := \{X \subseteq \mathbf{2^n} \mid X \models \phi\}$$
of an n-formula $\phi(p_{i_1}, \ldots, p_{i_n})$ is downwards closed and non-empty.

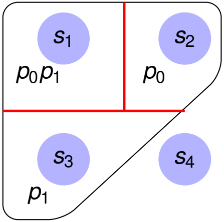
Let ∇_N be the family of all non-empty downwards closed collections of n-teams on N, i.e.
$$\nabla_N := \{\mathcal{K} \subseteq 2^{\mathbf{2^n}} \mid \mathcal{K} \neq \emptyset, \ \big(X \in \mathcal{K}, \ Y \subseteq X \Longrightarrow Y \in \mathcal{K}\big)\}.$$

Theorem (Huuskonen)

PD *is a maximal downwards closed logic, that is,*
$$\nabla_N = \{[\![\phi]\!] \mid \phi(p_{i_1}, \ldots, p_{i_n}) \textit{ is an n-formula of } \mathbf{PD}\}.$$

$\mathbf{PD}^{\vee}$ is also a maximal downwards closed logic.

An n-team is a finite set of valuations.



$$X = \{s_1, s_2, s_3\}$$

- s_1: $p_0 \wedge p_1$
- s_2: $p_0 \wedge \neg p_1$
- s_3: $\neg p_0 \wedge p_1$

$$Y \models (p_0 \wedge p_1) \otimes (p_0 \wedge \neg p_1) \otimes (\neg p_0 \wedge p_1) \iff Y \subseteq X$$

Lemma

Let X be an n-team on $N = \{i_1, \dots, i_n\}$. Let

$$\Theta_X := \bigotimes_{s \in X} (p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)}).$$

Then for any n-team Y,

$$Y \models \Theta_X \iff Y \subseteq X.$$

An n-formula corresponds to a non-empty downwards closed class of n-teams: $[\![\phi]\!] = \{X \mid X \models \phi\}$

Theorem (Normal Form)

Any n-formula $\phi(p_{i_1}, \dots, p_{i_n})$ of $\mathbf{PD}^{\vee}$ is semantically equivalent to a formula of the form

$$\bigvee_{i \in I} \Theta_{X_i},$$

for some finite set $\{X_i \mid i \in I\}$ of n-teams.

Axiomatizing $\mathbf{PD}^{\vee}$

In the following natural deduction system, one can derive the normal form for $\mathbf{PD}^{\vee}$ syntactically, namely:

Theorem (Normal Form)

Any $\mathbf{PD}^{\vee}$ *formula* $\phi(p_{i_1}, \dots, p_{i_n})$ *is provably equivalent to a formula of the form*

$$\bigvee_{i \in I} \Theta_{X_i},$$

for some finite set $\{X_i \mid i \in I\}$ *of n-teams.*

A natural deduction system for $\mathbf{PD}^{\vee}$

1. Dependence Atom Introduction:

$$\dfrac{\bigvee_{f\in 2^{\mathbf{2^k}}} \bigotimes_{s\in\mathbf{2^k}} \left(p_{j_0}^{s(j_0)} \wedge \dots \wedge \dots p_{j_{k-1}}^{s(j_{k-1})} \wedge p_{j_k}^{f(s)}\right)}{=\!(p_{j_0}, \dots, p_{j_{k-1}}, p_{j_k}),} \text{ (DepI)}$$

where $\mathbf{2^k}$ is the maximal team on $\{j_0, \dots, j_{k-1}\}$.

2. Dependence Atom Elimination:

$$\dfrac{=\!(p_{j_0}, \dots, p_{j_{k-1}}, p_{j_k})}{\bigvee_{f\in 2^{\mathbf{2^k}}} \bigotimes_{s\in\mathbf{2^k}} \left(p_{j_0}^{s(j_0)} \wedge \dots \wedge \dots p_{j_{k-1}}^{s(j_{k-1})} \wedge p_{j_k}^{f(s)}\right)} \text{ (DepE),}$$

where $\mathbf{2^k}$ is the maximal team on $\{j_0, \dots, j_{k-1}\}$.

3. Conjunction Introduction:

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}\ (\wedge\text{I})$$

4. Conjunction Elimination:

$$\frac{\phi \wedge \psi}{\phi}\ (\wedge\text{E}) \qquad \frac{\phi \wedge \psi}{\psi}\ (\wedge\text{E})$$

5. Intuitionistic Disjunction Introduction:

$$\frac{\phi}{\phi \vee \psi}\ (\vee\text{I}) \qquad \frac{\psi}{\phi \vee \psi}\ (\vee\text{I})$$

6. Intuitionistic Disjunction Elimination:

$$\frac{\phi \vee \psi \qquad \begin{matrix}[\phi]\\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\psi]\\ \vdots \\ \chi\end{matrix}}{\chi}\ (\vee\text{E})$$

7. Tensor Disjunction Introduction:

$$\frac{\phi}{\phi \otimes \psi}\,(\otimes\text{I}) \qquad \frac{\psi}{\phi \otimes \psi}\,(\otimes\text{I})$$

8. Weak Tensor Disjunction Elimination:

$$\frac{\phi \otimes \psi \qquad \begin{matrix}[\phi]\\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\psi]\\ \vdots \\ \chi\end{matrix}}{\chi}\,(\otimes\text{WE})$$

whenever χ does not contain dependence atoms or intuitionistic disjunction.

9. Tensor Disjunction Substitution:

$$\frac{\phi \otimes \psi \qquad \begin{matrix}[\psi]\\ \vdots \\ \chi\end{matrix}}{\phi \otimes \chi}\,(\otimes\text{Sub})$$

10. Commutative law and associative law of tensor disjunction

$$\frac{\phi \otimes \psi}{\psi \otimes \phi}\ (\mathrm{Com}\otimes) \qquad \frac{\phi \otimes (\psi \otimes \chi)}{(\phi \otimes \psi) \otimes \chi}\ (\mathrm{Ass}\otimes)$$

11. Atomic Excluded Middle:

$$\frac{}{p_i \otimes \neg p_i}\ (\mathrm{EM}_0)$$

12. Contradiction Elimination:

$$\frac{\phi \otimes (p_i \wedge \neg p_i)}{\phi}\ (\bot\mathrm{E})$$

13. Distributive Laws:

$$\frac{\phi \otimes (\psi \vee \chi)}{(\phi \otimes \psi) \vee (\phi \otimes \chi)}\ (\mathrm{Dstr} \otimes \vee) \qquad \frac{(\phi \otimes \psi) \vee (\phi \otimes \chi)}{\phi \otimes (\psi \vee \chi)}\ (\mathrm{Dstr} \otimes \vee\otimes)$$

$$\frac{\phi \wedge (\psi \otimes \chi)}{(\phi \wedge \psi) \otimes (\phi \wedge \chi)}\ (\mathrm{Dstr} \wedge \otimes)$$

Theorem (Normal Form)

Any $\mathbf{PD}^{\vee}$ *formula* $\phi(p_{i_1},\dots,p_{i_n})$ *is provably equivalent to a formula of the form*

$$\bigvee_{i\in I}\Theta_{X_i},$$

for some finite set $\{X_i \mid i\in I\}$ *of n-teams, where*

$$\Theta_X := \bigotimes_{s\in X}(p_{i_1}^{s(i_1)}\wedge\cdots\wedge p_{i_n}^{s(i_n)}).$$

Theorem (Completeness Theorem)

For any $\mathbf{PD}^{\vee}$ *formulas* ϕ *and* ψ,

$$\phi\models\psi \Longrightarrow \phi\vdash\psi.$$

Proof. (idea) Use the normal form. □

Axiomatizing **PD**

Crucial theorem in the proof of $\mathbf{PD}^{\vee}$ completeness theorem:

Theorem (Normal Form)

Any $\mathbf{PD}^{\vee}$ formula $\phi(p_{i_1}, \ldots, p_{i_n})$ is provably equivalent to a formula of the form

$$\bigvee_{i \in I} \Theta_{X_i},$$

for some finite set $\{X_i \mid i \in I\}$ of n-teams, where $\Theta_X := \bigotimes_{s \in X}(p_{i_1}^{s(i_1)} \wedge \cdots \wedge p_{i_n}^{s(i_n)})$.

However, in **PD**, we do not have intuitionistic disjunction $\vee$.

For a formula of **PD**, dependence atoms are the only source of the (potential) intuitionistic disjunction in the above normal form:

$$=\!(p_0, p_1) \dashv\vdash \bigvee_{f \in 2^{\mathbf{2}^{\mathbf{1}}}} \bigotimes_{f \in \mathbf{2}^{\mathbf{1}}} (p_0^{s(0)} \wedge p_1^{f(s)}),$$

where $\mathbf{2}^{\mathbf{1}}$ is the maximal 1-team on $\{0\}$.

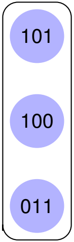
The idea is that we push the potential intuitionistic disjunctions of a formula of **PD** *in one step* to the front, and obtain *in effect* the above normal form *without actually using intuitionistic disjunction*.

Approximating dependence atoms

For a fixed n-team X such that

$$X \models (\,{=}(p_0, p_1) \wedge \phi) \otimes \psi,$$

the *way* that the dependence atom ${=}(p_0, p_1)$ is satisfied is fixed:



Therefore, for this fixed n-team X,

$$(\,{=}(p_0, p_1) \wedge \phi) \otimes \psi$$

is equivalent to

$$\Big(\big((p_0 \wedge \neg p_1) \otimes (\neg p_0 \wedge p_1)\big) \wedge \phi\Big) \otimes \psi$$

Hence

$$(\,{=}(p_0, p_1) \wedge \phi) \otimes \psi \equiv \bigvee_{f \in 2^{\mathbf{2^1}}} \Big(\big(\bigotimes_{f \in \mathbf{2^1}} (p_0^{s(0)} \wedge p_1^{f(s)}) \wedge \phi\big) \otimes \psi\Big),$$

where $\mathbf{2^1}$ is the maximal 1-team on $\{0\}$.

Approximating dependence atoms

Let ϕ be a formula of **PD**. List all occurrences of all dependence atoms in ϕ:

$$=\!(p_{j_0^1},\ldots,p_{j_{k_1}^1}),\ldots,=\!(p_{j_0^c},\ldots,p_{j_{k_c}^c}).$$

For each of the above dependence atom, pick a function $f_\xi:\mathbf{2}^{\mathbf{k}_\xi}\to 2$ that "realizes" the dependence atom. An *approximation sequence* of ϕ is a sequence $\Omega=\langle f_1,\ldots,f_c\rangle$ of such functions.

Replace inductively each occurrence of a dependence atom in ϕ by an approximation given by the approximation sequence to obtain a formula ϕ^*_Ω without dependence atoms:

- $p^*_{\langle\rangle} := p$ and $(\neg p)^*_{\langle\rangle} := \neg p$;
- $(=\!(p_{j_0^\xi},\ldots,p_{j_{k_\xi}^\xi}))^*_{\langle f_\xi\rangle} := \bigotimes_{s\in\mathbf{2}^{\mathbf{k}}}(p_{j_0}^{s(j_0)}\wedge\cdots\wedge p_{j_{k-1}}^{s(j_{k-1})}\wedge p_{j_k}^{f_\xi(s)})$;
- $(\psi\wedge\chi)^*_\Omega := \psi^*_{\Omega^0}\wedge\chi^*_{\Omega^1}$;
- $(\psi\otimes\chi)^*_\Omega := \psi^*_{\Omega^0}\otimes\chi^*_{\Omega^1}$.

Lemma (Weak Normal Form)

Let ϕ be a **PD** *n-formula and Λ the set of all its approximation sequences. Then*

$$\phi \equiv \bigvee_{\Omega \in \Lambda} \phi^*_{\Omega}.$$

In the natural deduction system for **PD**, we derive *in effect* that $\phi \dashv\vdash \bigvee_{\Omega \in \Lambda} \phi^*_{\Omega}$.

We had the introduction and elimination rules for intuitionistic disjunction:

$$\frac{\phi}{\phi \vee \psi}\ (\vee\text{I}) \qquad \frac{\phi \vee \psi \qquad \begin{matrix}[\phi]\\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\psi]\\ \vdots \\ \chi\end{matrix}}{\chi}\ (\vee\text{E})$$

These two rules are now replaced by the approximation elimination and approximation transition rules:

$$\frac{\phi^*_{\Omega_i}}{\phi}\ (\text{ApE}) \qquad \frac{\dfrac{\begin{matrix}[\phi^*_{\Omega_0}]\\ \vdots\end{matrix}}{\psi} \ \cdots \quad \cdots \ \dfrac{\begin{matrix}[\phi^*_{\Omega_m}]\\ \vdots\end{matrix}}{\psi} \qquad \phi}{\psi}\ (\text{ApTr})$$

where $\Lambda = \{\Omega_0, \ldots, \Omega_m\}$ is the set of all approximation sequences for ϕ.

Lemma (Weak Normal Form)

Let ϕ be a **PD** *n-formula and Λ the set of all its approximation sequences. Then*

$$\phi \equiv \bigvee_{\Omega \in \Lambda} \phi^*_\Omega.$$

Theorem (Completeness)

For any **PD** *formulas ϕ and ψ,*

$$\phi \models \psi \Longrightarrow \phi \vdash \psi.$$

Proof. (idea) Apply Weak Normal Form, (ApE), (ApTr) and the completeness theorem for **PD**$^\vee$. □

Armstrong Axioms are derivable in this system:

- $\vdash =(p, p)$
- $=(p, q, r) \vdash =(q, p, r)$
- $=(q, r) \vdash =(p, q, r)$
- $=(p, q), =(q, r) \vdash =(p, r)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall f : \mathbf{2^1} \to 2 \qquad [(\;=\!(p_1, p_2)\;)^*_{\langle f \rangle}]}{\bigotimes_{s \in \mathbf{2^1}} (p_1^{s(1)} \wedge p_2^{f(s)})} \qquad \dfrac{}{p_0 \otimes \neg p_0}\,(\mathrm{EM}_0)}{\left(\bigotimes_{s \in \mathbf{2^1}} (p_1^{s(1)} \wedge p_2^{f(s)})\right) \wedge (p_0 \otimes \neg p_0)}\,(\wedge\mathrm{I})}{\bigotimes_{s \in \mathbf{2^2}} ((p_0 \wedge p_1^{s(1)} \wedge p_2^{f(s)}) \otimes (\neg p_0 \wedge p_1^{s(1)} \wedge p_2^{f(s)}))}\,(\mathrm{Dstr}\ \wedge\otimes)}{\bigotimes_{s \in \mathbf{2^2}} (p_0^{s(0)} \wedge p_1^{s(1)} \wedge p_2^{g(s)})}}{=\!(p_0, p_1, p_2)}\,(\mathrm{DepSI}) \qquad =\!(p_1, p_2)}{=\!(p_0, p_1, p_2)}\,(\mathrm{ApTr})
$$

where $\mathbf{2^1}$, $\mathbf{2^2}$ are the maximal 1-team on $\{1\}$ and 2-team on $\{0, 1\}$, respectively, and $g : \mathbf{2^2} \to 2$ is defined as

$$g(s) = f(s \upharpoonright \{1\}).$$