KNAW Master Class

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Modal Dependence Logic

Tutorial

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History

Modal dependence logic was preceded by IF modal logic. There are several different versions defined by Tulenheimo 03,04 and Tulenheimo, Sevenster 06,07.

In IF modal logic, diamonds can be slashed by boxes that precede them: $\Box_1(\Diamond_2/\Box_1)\varphi$.

Modal dependence logic was introduced by Väänänen 09. The idea is quite different than in IF modal logic: dependencies are not between states, but between truth values of propositions.

The complexity of modal dependence logic was first studied by Sevenster 09. There has been a lot of research on complexity questions recently; e.g. Lohmann, Vollmer 10.

A new theme that is emerging is the study of modal logics with other types of dependency atoms, like modal independence logic and modal inclusion logic.

Basic modal logic $\mathcal{ML}$: Syntax

Let Φ be a set of proposition symbols. The set of $\mathcal{ML}(\Phi)$ -formulas is defined by the following grammar:

$$\varphi ::= p \mid \neg\, p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Box\varphi \mid \Diamond\varphi,$$

where $p \in \Phi$.

Note that formulas are assumed to be in *negation normal form*: negations may occur only in front of atomic formulas.

Basic modal logic $\mathcal{ML}$: Semantics

A *Kripke-model* for Φ is a triple $M = (W, R, V)$, where

- $W \neq \emptyset$ is the set of *states* (or possible worlds),
- $R \subseteq W \times W$ is the *accessibility relation*, and
- $V : \Phi \to \mathcal{P}(W)$ is the *valuation*.

A *team* on M is a subset $T \subseteq W$.

Basic modal logic $\mathcal{ML}$: Semantics

Kripke-semantics for $\mathcal{ML}$:

- $M, w \models p \Leftrightarrow w \in V(p)$
- $M, w \models \neg p \Leftrightarrow w \notin V(p)$
- $M, w \models \varphi \wedge \psi \Leftrightarrow M, w \models \varphi$ and $M, w \models \psi$
- $M, w \models \varphi \vee \psi \Leftrightarrow M, w \models \varphi$ or $M, w \models \psi$
- $M, w \models \Box\varphi \Leftrightarrow M, v \models \varphi$ for all v s.t. wRv
- $M, w \models \Diamond\varphi \Leftrightarrow M, v \models \varphi$ for some v s.t. wRv

Basic modal logic $\mathcal{ML}$: Semantics

Kripke-semantics/team semantics for $\mathcal{ML}$:

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Basic modal logic $\mathcal{ML}$: Semantics

Kripke-semantics/team semantics for $\mathcal{ML}$:

- $M, T \models p \quad \Leftrightarrow T \subseteq V(p)$
- $M, T \models \neg p \quad \Leftrightarrow T \cap V(p) = \emptyset$
- $M, w \models \varphi \wedge \psi \Leftrightarrow M, w \models \varphi$ and $M, w \models \psi$
- $M, w \models \varphi \vee \psi \Leftrightarrow M, w \models \varphi$ or $M, w \models \psi$
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- $M, T \models \varphi \vee \psi \Leftrightarrow M, S \models \varphi$ and $M, S' \models \psi$ for some $S \cup S' = T$
- $M, w \models \Box\varphi \quad \Leftrightarrow M, v \models \varphi$ for all v s.t. wRv
- $M, w \models \Diamond\varphi \quad \Leftrightarrow M, v \models \varphi$ for some v s.t. wRv

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- $M, T \models \Box\varphi \Leftrightarrow M, S \models \varphi$ for $S = \{v \in W \mid \exists w \in T : wRv\}$
- $M, w \models \Diamond\varphi \Leftrightarrow M, v \models \varphi$ for some v s.t. wRv

Basic modal logic $\mathcal{ML}$: Semantics

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- $M, T \models \Box\varphi \Leftrightarrow M, S \models \varphi$ for $S = \{v \in W \mid \exists w \in T : wRv\}$
- $M, T \models \Diamond\varphi \Leftrightarrow M, S \models \varphi$ for some S s.t. $\forall w \in T \exists v \in S : wRv$

Basic modal logic $\mathcal{ML}$: Semantics

Kripke-semantics/team semantics for $\mathcal{ML}$:

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- $M, T \models \neg p \quad \Leftrightarrow T \cap V(p) = \emptyset$
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- $M, T \models \varphi \vee \psi \Leftrightarrow M, S \models \varphi$ and $M, S' \models \psi$ for some $S \cup S' = T$
- $M, T \models \Box\varphi \quad \Leftrightarrow M, S \models \varphi$ for $S = \{v \in W \mid \exists w \in T : wRv\}$
- $M, T \models \Diamond\varphi \quad \Leftrightarrow M, S \models \varphi$ for some S s.t.
  $\forall w \in T \exists v \in S : wRv$ and $\forall v \in S \exists W \in T : wRv$

The idea behind team semantics is that a team T satisfies an $\mathcal{ML}$-formula φ iff all states $w \in T$ satisfy φ:

Theorem (Flatness property of $\mathcal{ML}$)

For all $\varphi \in \mathcal{ML}$,

$$M, T \models \varphi \;\Leftrightarrow\; M, w \models \varphi \text{ for all } w \in T.$$

In particular $M, \{w\} \models \varphi \;\Leftrightarrow\; M, w \models \varphi$.

Note that it also follows that every $\mathcal{ML}$-formula φ is *downwards closed*:

If $M, T \models \varphi$, then $M, S \models \varphi$ for all $S \subseteq T$.

Modal dependence logic $\mathcal{MDL}$

Let Φ be a set of proposition symbols. The set of $\mathcal{MDL}(\Phi)$ -formulas is defined by the following grammar:

$$\begin{array}{rcl}\varphi & ::= & p \mid \neg p \mid =(p_1, \ldots, p_n, q) \mid \\ & & (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \Diamond\varphi,\end{array}$$

where $p, p_1, \ldots, p_n, q \in \Phi$.

The propositional dependence atom $=(p_1, \ldots, p_n, q)$ says that the truth value of q is determined by the truth values of $p_1, \ldots, p_n$:

- $M, T \models =(p_1, \ldots, p_n, q) \quad \Leftrightarrow$
$$\forall v, w \in T : \bigwedge_{1 \leqslant i \leqslant n} (M, v \models p_i \Leftrightarrow M, w \models p_i) \Rightarrow (M, v \models q \Leftrightarrow M, w \models q)$$

Basic properties of $\mathcal{MDL}$

Theorem (Downwards closure for $\mathcal{MDL}$)

Let φ be a formula of $\mathcal{MDL}$.

$$\text{If } M, T \models \varphi, \text{ then } M, S \models \varphi \text{ for all } S \subseteq T.$$

Proof

By induction. The cases for $p, \neg p$ and $\varphi \wedge \psi$ are easy.

Assume $M, T \models \varphi \vee \psi$ and $S \subseteq T$. There are T', T'' s.t. $T = T' \cup T''$, $M, T' \models \varphi$ and $M, T'' \models \psi$.
Define $S' = S \cap T'$ and $S'' = S \cap T''$. Then $S = S' \cup S''$, and by IH, $M, S' \models \varphi$ and $M, S'' \models \psi$. Hence $M, S \models \varphi \vee \psi$.

Basic properties of $\mathcal{MDL}$

Assume $M, T \models \Diamond\varphi$ and $S \subseteq T$. There is T' s.t. $\forall w \in T \exists v \in T'$ s.t. wRv and $M, T' \models \varphi$.
Let $S' = \{v \in T' \mid \exists w \in S : wRv\}$. Then $\forall w \in S \exists v \in S'$ s.t. wRv, $\forall v \in S' \exists w \in S$ s.t. wRv, and by IH $M, S' \models \varphi$. Hence $M, S \models \Diamond\varphi$.

For the case $\Box\varphi$ it suffices to observe that if $S \subseteq T$, then

$$\{v \in W \mid \exists w \in S : wRv\} \subseteq \{v \in W \mid \exists w \in T : wRv\}.$$

For $=(p_1, \ldots, p_n, q)$ observe that if $S \subseteq T$, and S contains states v and$w$ that violate the truth condition of $=(p_1, \ldots, p_n, q)$, then the same holds for T. $\Box$

Complexity of $\mathcal{MDL}$

Theorem (Sevenster 09)

The satisfiability problem for $\mathcal{MDL}$ is $\mathrm{NEXPTIME}$-complete.

Lohmann and Vollmer 10 proved that the satisfiability problem remains $\mathrm{NEXPTIME}$-complete even if disjunction is dropped from $\mathcal{MDL}$ (poor man's $\mathcal{MDL}$).

In fact, they classified the complexity of all fragments of $\mathcal{MDL}$ obtained by restricting the connectives allowed.

Propositional dependence logic $\mathcal{PDL}$: Syntax

To understand the expressive power of $\mathcal{MDL}$, we study first its restriction to propositional formulas.

Let Φ be a set of proposition symbols. The set of $\mathcal{PDL}(\Phi)$-formulas is defined by the following grammar:

$$\begin{aligned} \varphi \quad ::= \quad & p \mid \neg p \mid =(p_1, \ldots, p_n, q) \mid \\ & (\varphi \vee \varphi) \mid (\varphi \wedge \varphi), \end{aligned}$$

where $p, p_1, \ldots, p_n, q \in \Phi$.

Propositional dependence logic $\mathcal{PDL}$: Semantics

A *(truth value) assignment* for Φ is a function $s : \Phi \to \{\bot, \top\}$. The semantics of $\mathcal{PDL}$ is defined on teams, that are just sets of assigments for Φ.

- $X \models p \quad \Leftrightarrow s(p) = \top$ for all $s \in X$
- $X \models \neg p \quad \Leftrightarrow s(p) = \bot$ for all $s \in X$
- $X \models \varphi \wedge \psi \Leftrightarrow X \models \varphi$ and $X \models \psi$
- $X \models \varphi \vee \psi \Leftrightarrow Y \models \varphi$ and $Z \models \psi$ for some $Y \cup Z = X$
- $X \models {=}(p_1, \ldots, p_n, q) \;\Leftrightarrow\; \bigwedge_i (s(p_i) = t(p_i)) \Rightarrow s(q) = t(q)$ holds for all $s, t \in X$

Note that there are 2^n assignments, and 2^{2^n} teams, where $n = |\Phi|$.

Intuitionistic disjunction

Add a different version of disjunction $\varovee$ to propositional (dependence) logic with the semantics:

- $X \models \varphi \varovee \psi \;\Leftrightarrow\; X \models \varphi$ or $X \models \psi$

Let $\mathcal{PL}(\varovee)$ be the the logic obtained from $\mathcal{PDL}$ by removing dependence atoms and adding $\varovee$.

Dependence atoms are definable in $\mathcal{PL}(\varovee)$ (Väänänen 09):

$$\models \,=\!(p_1, \ldots, p_n, q) \;\Leftrightarrow\; \bigvee_{s \in F} (\theta_s \wedge (q \varovee \neg q)),$$

where F is the team of all $\{p_1, \ldots, p_n\}$-assignment, and θ_s is the formula $\bigwedge_i p_i^{s(p_i)}$, where $p_i^{\bot} = \neg p_i$ and $p_i^{\top} = p_i$.

It is easy to prove by induction that for every $\mathcal{PDL}$-formula there is an equivalent $\mathcal{PL}(\varovee)$-formula. Thus, $\mathcal{PDL} \leqslant \mathcal{PL}(\varovee)$.

Intuitionistic disjunction: Completeness

We can prove a much stronger result: $\mathcal{PL}(\varovee)$ is *complete* with respect to downwards closed properties of teams.

Definition

- A *property* of teams is any set $\mathcal{P}$ of teams (for a fixed Φ).
- $\mathcal{P}$ is *downwards closed* if $X \in \mathcal{P}$ and $Y \subseteq X$ implies $Y \in \mathcal{P}$.
- A formula φ *defines* $\mathcal{P}$ if $\mathcal{P} = \{X \mid X \models \varphi\}$.

Theorem (Yang 14)

Every downwards closed property of teams is definable in the logic $\mathcal{PL}(\varovee)$.

Intuitionistic disjunction: Completeness

Proof.

For each Φ-assignment s, let θ_s be the formula $\bigwedge_{p\in\Phi} p^{s(p)}$.
Clearly $Y \models \theta_s$ iff $Y \subseteq \{s\}$.

For each Φ-team X, let ψ_X be the formula $\bigvee_{s\in X} \theta_s$.
Then we have $Y \models \psi_X$ iff $Y \subseteq X$.

Finally, if $\mathcal{P}$ is a downwards closed property of teams,
let $\varphi_{\mathcal{P}}$ be the formula $\bigovee_{X\in\mathcal{P}} \psi_X$.

Now we have

$$Y \models \varphi_{\mathcal{P}} \quad\Leftrightarrow\quad \exists X \in \mathcal{P} : Y \subseteq X \quad\Leftrightarrow\quad Y \in \mathcal{P}.$$

□

Remark: The proof gives a normal form for $\mathcal{PL}(\ovee)$-formulas φ:

$$\varphi \;\Leftrightarrow\; \bigovee_{i\in I} \psi_J, \text{ where each } \psi_J \in \mathcal{PL} \text{ and is in DNF.}$$

$\mathcal{PDL}$ and intuitionistic disjunction

Since $\mathcal{PDL}$ is downwards closed, we have another proof for the fact $\mathcal{PDL} \leqslant \mathcal{PL}(\varovee)$.
Note that both methods lead to an exponential blow-up in the size of formulas. This not accidental:

Theorem (Luosto 14)

If $\varphi \in \mathcal{PL}(\varovee)$ is equivalent with $=(p_1, \ldots, p_n, q)$, then φ contains at least 2^n occurrences of $\varovee$.

On the other hand, there is also a translation in the opposite direction:

Theorem (Huuskonen, Yang 14)

Every downwards closed property of teams is definable in $\mathcal{PDL}$. Hence $\mathcal{PL}(\varovee) \leqslant \mathcal{PDL}$.

$\mathcal{PDL}$ and intuitionistic disjunction

Proof.

Consider the formula $\gamma_\Phi := \bigwedge_{p\in\Phi} {=}(p)$. It says that every $p \in \Phi$ has constant truth value, whence $X \models \gamma_\Phi$ iff $|X| \leqslant 1$.

Define recursively

$$\gamma_\Phi^1 := \gamma_\Phi, \quad \gamma_\Phi^{k+1} := (\gamma_\Phi^k \vee \gamma_\Phi).$$

Then we have for all k, $X \models \gamma_\Phi^k$ iff $|X| \leqslant k$.

If Y is a team such that $|Y| = k+1$, we let $\chi_Y := \psi_Z \vee \gamma_\Phi^k$, where Z is the complement of Y. Now

$$\begin{aligned} X \models \chi_Y \quad &\Leftrightarrow \quad X \cap Z \neq \emptyset \text{ or } (X \cap Z = \emptyset \text{ and } |X| \leqslant k) \\ &\Leftrightarrow \quad Y \nsubseteq X. \end{aligned}$$

Finally, if $\mathcal{P}$ is a downwards closed property of teams, then the formula $\eta_\mathcal{P} := \bigwedge_{Y\notin\mathcal{P}} \chi_Y$ defines it. □

Expressive power $\mathcal{ML}(\owedge)$

Let $\mathcal{ML}(\owedge)$ be the extension of $\mathcal{ML}$ with intuitionistic disjunction. We lift the completeness result from $\mathcal{PL}(\owedge)$ to $\mathcal{ML}(\owedge)$.

First we recall a characterization for the expressive power of $\mathcal{ML}$ with respect to Kripke-semantics.

Theorem (van Benthem, Gabbay)

Assume that Φ is finite. A class $\mathcal{K}$ of pointed Kripke models (M, w) is definable in $\mathcal{ML}$ if and only if there is k such that $\mathcal{K}$ is closed under k-bisimulation:

$$\text{if } (M, w) \in \mathcal{K} \text{ and } (M, w) \sim_k (N, v), \text{ then } (N, v) \in \mathcal{K}.$$

Remark: The defining formula of $\mathcal{K}$ can be chosen to be a disjunction of Hintikka-formulas $\alpha^k_{M,w}$; the formula $\alpha^k_{M,w}$ describes the pair (M, w) up to k-bisimilarity.

Expressive power $\mathcal{ML}(\varovee)$

Definition

- A property of modal teams is a class $\mathcal{K}$ of pairs (M, T), where T is a Φ-team in M for a fixed Φ.
- $\mathcal{K}$ is downwards closed if $(M, T) \in \mathcal{K}$ and $S \subseteq T$ implies $(M, S) \in \mathcal{K}$.
- $\mathcal{K}$ is closed under k-bisimulation if $(M, T) \in \mathcal{K}$ implies that $(M, T^*) \in \mathcal{K}$ for $T^* := \{v \in W \mid \exists w \in T : (M, v) \sim_k (M, w)\}$.
- A formula φ defines $\mathcal{K}$ if $\mathcal{K} = \{(M, T) \mid M, T \models \varphi\}$.

Theorem (Virtema 14)

Assume that Φ is finite. A property $\mathcal{K}$ of modal teams is definable in $\mathcal{ML}(\varovee)$ if and only if $\mathcal{K}$ is downwards closed and closed under k-bisimulation for some k.

Expressive power $\mathcal{MDL}$

Since $\mathcal{PDL} \equiv \mathcal{PL}(\varovee)$, it is natural to ask, whether the modal counterpart of this equivalence holds.
It is not difficult to prove the first direction:

Theorem (Ebbing, H, Meier, Müller, Virtema, Vollmer 13)
$\mathcal{MDL} \leqslant \mathcal{ML}(\varovee)$.

However, the converse is not true:

Example
There is no formula $\varphi \in \mathcal{MDL}$ that is equivalent with $\Diamond p \varovee \Box\neg p$.
Proof idea: $\mathcal{MDL}$ collapses to $\mathcal{ML}$ on the Kripke model $M = (\{a, b\}, \{(b, b)\}, p \mapsto \{a, b\})$. On the other hand, $\Diamond p \varovee \Box\neg p$ is not expressible in $\mathcal{ML}$ on M, since it is not flat on M.

Extended modal dependence logic $\mathcal{EMDL}$

What is missing from $\mathcal{MDL}$? The counterexample gives a clue: the formula $\Diamond p \otimes \Box \neg p$ says that the truth value of $\Diamond p$ is constant. That is, $\Diamond p \otimes \Box \neg p$ is equivalent to $=(\Diamond p)$.

Let Φ be a set of proposition symbols. The set of $\mathcal{EMDL}(\Phi)$-formulas is defined by the following grammar:

$$\begin{aligned} \varphi \quad ::= \quad & p \mid \neg\, p \mid =(\alpha_1, \ldots, \alpha_n, \beta) \mid \\ & (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \Diamond\varphi, \end{aligned}$$

where $p \in \Phi$ and $\alpha_1, \ldots, \alpha_n, \beta \in \mathcal{ML}$.

The semantics of $=(\alpha_1, \ldots, \alpha_n, \beta)$ is defined in the same way as for $=(p_1, \ldots, p_n, q)$.

Remark: We do not allow nested dependence atoms!

Expressive power of $\mathcal{EMDL}$

Since $=(\Diamond p)$ is not expressible in $\mathcal{MDL}$, $\mathcal{EMDL}$ is a proper extension of $\mathcal{MDL}$. It is straightforward to prove that $\mathcal{EMDL}$ is still contained in $\mathcal{ML}(\varovee)$:

Theorem (Ebbing, H, Meier, Müller, Virtema, Vollmer 13)
$\mathcal{MDL} < \mathcal{EMDL} \leqslant \mathcal{ML}(\varovee)$.

Using the method of Huuskonen, we can now prove that $\mathcal{ML}(\varovee)$ is contained $\mathcal{EMDL}$:

Theorem (H, Luosto, Virtema 14)
A property $\mathcal{K}$ of modal teams is definable in $\mathcal{EMDL}$ if and only if $\mathcal{K}$ is downwards closed and closed under k-bisimulation for some k. Thus, $\mathcal{EMDL} \equiv \mathcal{ML}(\varovee)$.

Complexity of $\mathcal{EMDL}$

Although $\mathcal{EMDL}$ is a proper extension of $\mathcal{MDL}$, it has the same complexity:

Theorem (Ebbing, H, Meier, Müller, Virtema, Vollmer 13)

The satisfiability problem for $\mathcal{EMDL}$ is $\mathrm{NEXPTIME}$-complete.

On the other hand, $\mathcal{ML}(\varovee)$ is less complex than $\mathcal{EMDL}$:

Theorem (Sevenster 09)

The satisfiability problem for $\mathcal{ML}(\varovee)$ is PSPACE-complete.

This is explained by the fact that there is an exponential blow-up in translating from $\mathcal{EMDL}$ to $\mathcal{ML}(\varovee)$.

Modal inclusion logic $\mathcal{MINC}$

Modal inclusion logic is the extension of $\mathcal{ML}$ with modal inclusion atoms $\alpha_1 \ldots \alpha_n \subseteq \beta_1 \ldots \beta_n$.

Here $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are arbitrary $\mathcal{ML}$-formulas.
(We adopt the "extended" framework from the beginning.)

The semantics of modal inclusion atoms is defined as follows:

- $M, T \models \vec{\alpha} \subseteq \vec{\beta} \quad \Leftrightarrow \quad \forall v \in T \exists w \in T :$
$$\bigwedge_i (M, v \models \alpha_i \Leftrightarrow M, w \models \beta_i),$$

  where $\vec{\alpha} = \alpha_1 \ldots \alpha_n$ and $\vec{\beta} = \beta_1 \ldots \beta_n$.

Modal inclusion logic $\mathcal{MINC}$

The formulas of $\mathcal{MINC}$ are not downwards closed in general. However, $\mathcal{MINC}$ has another useful closure property:

Theorem (Union closure for $\mathcal{MINC}$)

Let $\varphi \in \mathcal{MINC}(\Phi)$. If $M, T_1 \models \varphi$ and $M, T_2 \models \varphi$, then $M, T_1 \cup T_2 \models \varphi$.

Proof

By induction on φ. We consider the cases of $\varphi \vee \psi$ and $\Diamond\varphi$.

If $M, T_i \models \varphi \vee \psi$ for $i \in \{1,2\}$, then there are T_1', T_1'', T_2', T_2'' s.t. $T_i = T_i' \cup T_i''$, $M, T_i' \models \varphi$, and $M, T_i'' \models \psi$ for $i \in \{1,2\}$. By IH, $M, T_1' \cup T_2' \models \varphi$ and $M, T_1'' \cup T_2'' \models \psi$. Hence, $M, (T_1' \cup T_2') \cup (T_1'' \cup T_2'') \models \varphi \vee \psi$.

Modal inclusion logic $\mathcal{MINC}$

If $M, T_1 \models \Diamond\varphi$ and $M, T_2 \models \Diamond\varphi$, then there are T_1', T_2' s.t. $M, T_i' \models \varphi$, $\forall w \in T_i \exists v \in T_i' : wRv$ and $\forall v \in T_i' \exists w \in T_i : wRv$. By IH, $M, T_1' \cup T_2' \models \varphi$, and clearly $\forall w \in T \exists v \in T' : wRv$ and $\forall v \in T' \exists w \in T : wRv$ for $T = T_1 \cup T_2$ and $T' = T_1' \cup T_2'$. □

Theorem

(a) Let $\varphi \in \mathcal{MINC}$, and let T be a team on a Kripke model M. There exists a maximal subteam $S \subseteq T$ such that $M, S \models \varphi$.

(b) Moreover, for each φ there is a PTIME algorithm A_φ such that on input (M, T), A_φ outputs the maximal subteam S.

Complexity of $\mathcal{MINC}$

Corollary (H, Meier, Vollmer 14)

$\mathcal{MINC} \leqslant \mathrm{PTIME}$; i.e., all $\mathcal{MINC}$-definable classes are PTIME-computable.

Using the algorithms A_φ, we can also prove that the satisfiability problem for $\mathcal{MINC}$ is less complex as that for $\mathcal{EMDL}$:

Theorem (H, Meier, Vollmer 14)

The satisfiability problem for $\mathcal{MINC}$ is in $\mathrm{EXPTIME}$.

Expressive power of $\mathcal{MINC}$

The expressive power of $\mathcal{MINC}$ can be analyzed in the same way as for $\mathcal{MDL}$ and $\mathcal{EMDL}$: Study first propositional inclusion logic $\mathcal{PINC}$, which is obtained by dropping the modal operators.

Theorem (H 14)

A property $\mathcal{P}$ of teams is definable in $\mathcal{PINC}$ if and only if $\mathcal{P}$ is closed under unions.

This result can probably be used as a starting point for a characterization of the expressive power of $\mathcal{MINC}$. However, this is work in progress at the moment.