Independence in model theory

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Model theoretic motivation

Classification of structures

- find a system of invariants that will determine the structure up to isomorphism

Examples:

- transcendence degree of algebraically closed fields of fixed characteristic
- Ulm invariants of countable Abelian groups

Shelah's Main Gap

Theorem (Shelah)

A countable first order theory T falls into one of two categories:

- *it is intractable, i.e. it has the maximal number of models in all sufficiently large cardinalities or*
- *every model of T is decomposed as a tree of countable models*

Independence calculus

- Shelah's proof builds on notions of independence and generation.
- Shelah used nonforking independence.
- Proving structure theorems for more general classes involve developing suitable independence notions.

Model theoretic definitions: type

Definition

When studying the class of models of a first order theory T a *type over A* is a set of formulas with parameters from A that is consistent with T.

We write $tp(\bar{a}/A)$ for the type over A realised by $\bar{a}$.

Model theoretic definitions: monster model $\mathfrak{M}$

In elementary model theory a monster model is just a large saturated model.

When studying nonelementary classes (with amalgamation), a monster model is a μ-universal, strongly μ-homogeneous model for a large enough μ.

Model-theoretic definitions: Galois-type

When working inside a strongly homogeneous monster model the type of $\bar{a}$ over A is the orbit of $\bar{a}$ under automorphisms of $\mathfrak{M}$ fixing A pointwise.

$$tp(a/A) = \{f(a) : f \in Aut(\mathfrak{M}/A)\}.$$

In the elementary case this coincides with the syntactic type.

Model theoretic definitions: Stability

Definition

A theory T is *stable in* λ if there are only λ types (n-types for some $n < \omega$) over parameter sets of size λ.

A theory T is *stable* if it is stable in some λ.

A theory T is *superstable* if it is stable from some κ onwards.

A theory T is ω-stable if it is stable in $\aleph_0$ (and then stable in all infinite cardinalities).

Forking

Definition

The formula $\varphi(\bar{x}, \bar{a})$ *divides* over A if there are $n < \omega$ and tuples $\bar{a}_i$, $i < \omega$, such that

1. $tp(\bar{a}/A) = tp(\bar{a}_i/A)$,
2. $\{\varphi(\bar{x}, \bar{a}_i) : i < \omega\}$ is n-inconsistent (i.e. every n-size subset is inconsistent wrt the theory)

Definition

$p = tp(\bar{a}/A)$ *forks* over A if there are formulas $\varphi_0(\bar{x}_0, \bar{a}_0), \ldots, \varphi_{n-1}(\bar{x}_{n-1}; \bar{a}_{n-1})$ with $\bar{x}_k \subseteq \bar{x}$ for $k < n$, such that

1. $p \vdash \bigvee_{k<n} \varphi_k(\bar{x}_k, \bar{a}_k)$,
2. $\varphi_k(\bar{x}_k, \bar{a}_k)$ divides over A for all $k < n$.

In stable theories we have nonforking independence:
We write $A \downarrow_B C$ if for all finite tuples $\bar{a}$ from A, $tp(\bar{a}/C)$ does not fork over B.

Properties of nonforking

If T is a stable first order theory, then nonforking has the following properties

1. **Invariance** If $\bar{a} \downarrow_A B$ and F is an isomorphism, then $f(\bar{a}) \downarrow_{f(A)} f(B)$.
2. **Monotonicity** If $A \subseteq B \subseteq C \subseteq D$ and $\bar{a} \downarrow_A D$ then $\bar{a} \downarrow_B C$.
3. **Transitivity** If $A \subseteq B \subseteq C$, $\bar{a} \downarrow_A B$ and $\bar{a} \downarrow_B C$ then $\bar{a} \downarrow_A C$.
4. **Symmetry** If $\bar{a} \downarrow_A \bar{b}$ then $\bar{b} \downarrow_A \bar{a}$.
5. **Existence/Extension** For any $\bar{a}$ and $A \subset B$ there is $\bar{b}$ satisfying $tp(\bar{a}/A)$ such that $\bar{b} \downarrow_A B$.

Properties of nonforking

6. **Finite character** If $\bar{a} \not\downarrow_A B$ and $A \subset B$ then there is a formula $\varphi(\bar{x}, \bar{b}) \in tp(a/B)$ such that no type containing $\varphi(\bar{x}, \bar{b})$ is independent over A.
7. **Local character** There is a cardinal $\kappa(T)$ such that for any $\bar{a}$ and B there is $A \subseteq B$ such that $|A| < \kappa(T)$ and $\bar{a} \downarrow_A B$.
8. **Reflexivity** If $A \subset B$, $\bar{b} \in B \backslash A$ and $tp(b/A)$ is not algebraic, then $\bar{b} \not\downarrow_A B$.
9. **Stationarity over models** If $\mathcal{A}$ is a model, $tp(\bar{a}/\mathcal{A}) = tp(\bar{b}/\mathcal{A})$, $\bar{a} \downarrow_{\mathcal{A}} B$ and $\bar{b} \downarrow_{\mathcal{A}} B$, then $tp(\bar{a}/B) = tp(\bar{b}/B)$.

Towards greater generality

Generalising the framework

- elementary classes
- elementary submodels of a strongly homogeneous model
- abstract elementary classes
- "elementary" classes wrt continuous logic
- metric abstract elementary classes

Results require various stability assumptions (ω-stable, superstable, stable, weakly stable, simple)
Study different notions of types

- types
- Lascar types
- Lascar strong types
- types in continuous logic

Independence in homogeneous model theory

When working in a stable homogeneous class, one can define independence based on strong splitting.

Definition

1. A type $tp(a/B)$ is said to *split strongly over* $A \subset B$ if there are $b, c \in B$ and an infinite sequence I, indiscernible over A, with $b, c \in I$ such that $tp(b/A \cup a) \neq tp(c/A \cup a)$.
2. $\kappa(\mathbb{K})$ denotes the least cardinal such that there are no a, b_i and c_i for $i < \kappa(\mathbb{K})$ such that $tp(a/\bigcup_{j \le i}(b_j \cup c_j))$ splits strongly over $\bigcup_{j<i}(b_j \cup c_j)$ for each $i < \kappa(\mathbb{K})$.
3. $a \downarrow_A B$ if there is $C \subseteq A$ of cardinality $< \kappa(\mathfrak{M})$ such that for all $D \supseteq A \cup B$ there is b with $tp(b/A \cup B) = tp(a/A \cup B)$ such that $tp(b/D)$ does not split strongly over C.

Independence in homogeneous model theory

Theorem (Hyttinen-Shelah)

In a simple stable homogeneous class $\downarrow$ satisfies

- *Monotonicity*
- *Extension*
- *Finite character*
- *Symmetry*
- *Transitivity*
- *Stationarity for Lascar strong types*

Independence in finitary abstract elementary classes

Definition

A class of structures of a fixed vocabulary $(\mathbb{K}, \preccurlyeq)$ is an *abstract elementary class* if

1. Both $\mathbb{K}$ and the binary relation $\preccurlyeq$ are closed under isomorphism.
2. If $\mathcal{A} \preccurlyeq \mathcal{B}$ then $\mathcal{A}$ is a substructure of $\mathcal{B}$.
3. $\preccurlyeq$ is a partial order on $\mathbb{K}$.
4. If $\langle \mathcal{A}_i : i < \delta \rangle$ is an $\preccurlyeq$-increasing chain, then
   1. $\bigcup_{i<\delta} \mathcal{A}_i \in \mathbb{K}$,
   2. for each $j < \delta$, $\mathcal{A}_j \preccurlyeq \bigcup_{i<\delta} \mathcal{A}_i$,
   3. if each $\mathcal{A}_i \preccurlyeq \mathcal{M} \in \mathbb{K}$ then $\bigcup_{i<\delta} \mathcal{A}_i \preccurlyeq \mathcal{M}$.
5. If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{K}$, $\mathcal{A} \preccurlyeq \mathcal{C}$, $\mathcal{B} \preccurlyeq \mathcal{C}$ and $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{A} \preccurlyeq \mathcal{B}$.
6. There is a Löwenheim-Skolem number $LS(\mathbb{K})$ such that if $\mathcal{A} \in \mathbb{K}$ and $B \subset \mathcal{A}$, then there is $\mathcal{A}' \in \mathbb{K}$ such that $B \subseteq \mathcal{A}' \preccurlyeq \mathcal{A}$ and $|\mathcal{A}'| = |B| + LS(\mathbb{K})$.

Independence in finitary abstract elementary classes

Definition

In a simple, superstable, finitary AEC one can define

$$\bar{a} \downarrow_A B$$

if there is a finite $E \subseteq A$ such that for all monster models $\mathfrak{M}'$ extending $\mathfrak{M}$ and $D \subset \mathfrak{M}'$ such that $A \cup B \subset D$ there is a monster model $\mathfrak{M}''$ extending $\mathfrak{M}$ and $b \in \mathfrak{M}''$ such that $t^w(b/AB) = t^w(a/AB)$ and $t^w(b/D)$ does not Lascar-split over E.

Independence in finitary abstract elementary classes

Theorem (Hyttinen-Kesälä)

If $(\mathbb{K}, \preccurlyeq)$ is a simple, superstable, finitary AEC with the Tarksi-Vaught property, then the relation $\downarrow$ has the following properties

- *Invariance*
- *Monotonicity*
- *Transitivity*
- *Symmetry*
- *Extension*
- *Finite character*
- *Local character*
- *Reflexivity*
- *Stationarity for Lascar types*

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Continuous logic

Continuous logic

- takes truth values in the interval $[0,1]$
- has continuous functions $[0,1]^n \to [0,1]$ as connectives
- has sup and inf as quantifiers

Continuous logic is used to study bounded metric structures.

Dividing

Definition

A type $p(x, B)$ *divides over* A if there exists an A-indiscernible sequence $(B_i : i < \omega)$ in $tp(B/A)$ such that $\bigcup_{i<\omega} p(x, B_i)$ is inconsistent with T.

Independence in continuous logic

In continuous logic independence is defined via dividing

$$A \downarrow_B C$$

if and only if $tp(A/BC)$ does not divide over B.

Theorem (Ben Yaacov, Berenstein, Henson, Usvyatsov)

If T is a continuous theory, stable in the metric sense (i.e. considering the density character of the type set), then $\downarrow$ satisfies

- *Invariance*
- *Symmetry*
- *Transitivity*
- *Finite character*
- *Extension*
- *Local character*
- *Stationarity over models*

Uniqueness

Well-behaved independence notions are unique.

Theorem

Assume $\mathfrak{M}$ is stable and strongly homogeneous. Then any independence notion satisfying

- *Invariance*
- *Monotonicity*
- *Existence/Extension*
- *Stationarity over $F^{\mathfrak{M}}_{\lambda(\mathfrak{M})}$-saturated models*

is unique over $F^{\mathfrak{M}}_{\lambda(\mathfrak{M})}$-saturated models (i.e. if we have two independence notions, $\downarrow$ and $\downarrow'$ satisfying the properties, $\bar{a} \downarrow_A B$ and A is $F^{\mathfrak{M}}_{\lambda(\mathfrak{M})}$-saturated, then $\bar{a} \downarrow'_A B$).

Uniqueness

Theorem (Hyttinen-Lessmann)

Suppose $\mathfrak{M}$ is a stable homogeneous monster model and there exists an independence relation satisfying:

- *Invariance*
- *Monotonicity*
- *Local character*
- *Finite character*
- *Symmetry*
- *Transitivity*
- *Extension*
- *Bounded extensions*

Then $\mathfrak{M}$ is superstable and $A \downarrow_C B$ if and only if for all finite $\bar{a}$ in A and $\bar{b}$ in B $tp(\bar{a}/C\bar{b})$ does not divide over A.

Uniqueness

Theorem (Hyttinen-Kesälä)

In a superstable finitary AEC any independence notion satisfying

- *Invariance*
- *Reflexivity*
- *Extension*
- *Stationarity*
- *Monotonicity*
- *Transitivity*

is unique.

Connection to probability

Theorem (Ben Yaacov)

1. *The class of probability algebras is axiomatisable in continuous logic.*
2. *Random variables are interpretable in the event space and vice versa.*
3. *Model theoretic independence coincides with probabilistic independence.*