Generalized atoms and quantifiers

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$\grave{\exists} x\ \acute{\exists} x\ \hat{\exists} x\ \ddot{\exists} x$

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1. We generalize the notion of a Lindström quantifier. The novel *minor quantifiers*, or *constructive quantifiers*, are motivated by team semantics as well as independently of team semantics.
2. We develop a team semantics for a logic with any minor quantifiers and *generalized atoms*.
3. We also devise a corresponding *game theoretic semantics*. The game theoretic semantics is fully symmetric (w.r.t. negation).
4. We use the resulting semantics by extending two-variable dependence logic with counting quantifiers. We show that the resulting logic DC^2 is $\mathrm{NEXPTIME}$-complete.

Double team semantics

$$
\begin{array}{lcl}
\mathfrak{A}, (U, V) \models R(x_1, ..., x_m) & \Leftrightarrow & \forall s \in U\big(\mathfrak{A}, s \models_{\mathrm{FO}} R(x_1, ..., x_m)\big) \text{ and} \\
 & & \forall s \in V\big(\mathfrak{A}, s \not\models_{\mathrm{FO}} R(x_1, ..., x_m)\big). \\
\mathfrak{A}, (U, V) \models \neg\varphi & \Leftrightarrow & \mathfrak{A}, (V, U) \models \varphi. \\
\mathfrak{A}, (U, V) \models (\varphi \vee \psi) & \Leftrightarrow & \mathfrak{A}, (T, V) \models \varphi \text{ and } \mathfrak{A}, (S, V) \models \psi \text{ for} \\
 & & \text{some } T, S \subseteq U \text{ such that } T \cup S = U.
\end{array}
$$

Double team semantics

Verify that half of the numbers in 6 are odd, i.e., show that

$$\{0, 1, 2, 3, 4, 5\} \models \text{half of the numbers are odd.}$$

It suffices to find a subset $S \subseteq 6$ containing half of the elements in 6 such that

1. each $x \in S$ is odd,
2. none of the elements x in $6 \setminus S$ is odd.

Double team semantics

$$\mathfrak{A}, (U, V) \models Qx\,\varphi$$

if and only if there exist functions $f : U \to Q^{\mathfrak{A}}$ and $g : V \to \overline{Q}^{\mathfrak{A}}$ such that

$$\mathfrak{A},\ \big(\, U[\,x/f\,] \cup V[\,x/g\,],\ \ U[\,x/f'\,] \cup V[\,x/g'\,]\,\big)\ \models\ \varphi.$$

Here $Q^{\mathfrak{A}}$ is a generalized quantifier restricted to $\mathfrak{A}$. For example, $Q^{\mathfrak{A}}$ could contain exactly all sets $B \subseteq A = Dom(\mathfrak{A})$ that cover half of the elements of $\mathfrak{A}$.

$\overline{Q}^{\mathfrak{A}}$ would then contain exactly all $B \subseteq A$ that do *not* cover half of the elements of $Dom(\mathfrak{A})$.

$U[\,x/f\,] = \{\ s[x/b] \mid s \in U,\ b \in f(s)\ \}.$
$U[\,x/f'\,] = \{\ s[x/b] \mid s \in U,\ b \in A \setminus f(s)\ \}.$

Generalized atoms

Let (Q, P) be a pair of generalized quantifiers of the type $(i_1, ..., i_k)$. The pair (Q, P) defines a generalized atom $\mathcal{A}$.

Define $\mathfrak{A}, (U, V) \models \mathcal{A}(\overline{x}_1, ..., \overline{x}_k)$ iff

$$\big(\mathrm{Rel}(U, \mathfrak{A}, \overline{x}_1), ..., \mathrm{Rel}(U, \mathfrak{A}, \overline{x}_k)\big) \in Q^{\mathfrak{A}},$$

and

$$\big(\mathrm{Rel}(V, \mathfrak{A}, \overline{x}_1), ..., \mathrm{Rel}(V, \mathfrak{A}, \overline{x}_k)\big) \in P^{\mathfrak{A}}.$$

$\mathrm{Rel}(U, \mathfrak{A}, \overline{x}_i)$ is the relation induced by the tuple $\overline{x}_i$ and team U.

Double team semantics

Theorem

For any formula without generalized atoms (but possibly containing generalized quantifiers), $\mathfrak{A}, (U, V) \models \varphi$ *if and only if we have* $\forall s \in U\big(\mathfrak{A}, s \models_{\mathrm{FO}} \varphi\big)$ *and* $\forall t \in V\big(\mathfrak{A}, t \not\models_{\mathrm{FO}} \varphi\big)$.

game theoretic semantics

Positions of game $G(\mathfrak{A}, U, V, \varphi)$ are tuples $(\mathfrak{A}, s, \#, \psi)$, where s is an assignment, $\# \in \{+, -\}$, and ψ is a subformula of φ. (Two identical but distinct subformulae are considered different subformulae.)

Two players, *interrogator* $\mathbb{I}$ and *agent* $\mathbb{A}$.

When $\# = +$, the player $\mathbb{A}$ wishes to verify ψ. When $\# = -$, the player $\mathbb{A}$ wishes to falsify ψ.

game theoretic semantics

In the beginning of a play of $G(\mathfrak{A}, U, V, \varphi)$, $\mathbb{I}$ pics a beginning position $(\mathfrak{A}, s, +, \varphi)$, where $s \in U$, or $(\mathfrak{A}, t, -, \varphi)$, where $t \in V$.

The game is rather similar to the FO semantic game, but the quantifier moves and winning condition differ.

In a position $(\mathfrak{A}, s, +, Qx\,\psi)$, the player $\mathbb{A}$ pics a set $B \in Q^{\mathfrak{A}}$. Then $\mathbb{I}$ pics an element $b \in B$ or an element $b' \in B \setminus Dom(\mathrm{A})$. In the first case, the play continues from $(\mathfrak{A}, s[x \mapsto b], +, \psi)$, and in the latter case from $(\mathfrak{A}, s[x \mapsto b'], -, \psi)$.

(If $Q^{\mathfrak{A}} = \emptyset$, the play ends.)

In a position $(\mathfrak{A}, s, -, Qx\,\psi)$, the player $\mathbb{A}$ pics a set $B \in \overline{Q}^{\mathfrak{A}}$. Then $\mathbb{I}$ pics an element $b \in B$ or an element $b' \in B \setminus Dom(\mathrm{A})$. In the first case, the play continues from $(\mathfrak{A}, s[x \mapsto b], +, \psi)$, and in the latter case from $(\mathfrak{A}, s[x \mapsto b'], -, \psi)$.

In a position $(\mathfrak{A}, s, \#, \neg\psi)$, the symbol $\#$ changes (from $+$ to $-$, or vice versa). The play continues from $(\mathfrak{A}, s, \#, \psi)$.

game theoretic semantics

Moves in a position $(\mathfrak{A}, s, +, \psi \vee \chi)$ depend on the type of disjunction used. Consider the semantics $(U, V) \models \psi \vee \chi$ iff $(S, V) \models \psi$ and $(T, V) \models \chi$ for S, T such that $S \cup T$ and $S \cap T = \emptyset$.

Then $\mathbb{A}$ pics one of the positions $(\mathfrak{A}, s, +, \chi)$ and $(\mathfrak{A}, s, +, \psi)$, and the play continues from there. In a position $(\mathfrak{A}, s, -, \psi \vee \chi)$, the player $\mathbb{I}$ pics one of the positions $(\mathfrak{A}, s, -, \chi)$ and $(\mathfrak{A}, s, -, \psi)$.

In an end position $(\mathfrak{A}, s, +, \psi)$, where ψ is a first-order atom, the player $\mathbb{A}$ *survives* the play iff $\mathfrak{A}, s \models \psi$.

In an end position $(\mathfrak{A}, s, -, \psi)$, where ψ is a first-order atom, the player $\mathbb{A}$ *survives* the play iff $\mathfrak{A}, s \not\models \psi$.

In an end position $(\mathfrak{A}, s, \#, \psi)$, where ψ is a generalized atom, the player $\mathbb{A}$ *survives* the play.

game theoretic semantics

$\mathbb{A}$ has a *uniform survival strategy* if his strategy guarantees survival, and for each generalized atom ψ the following holds.

Uniformity condition:

(positive end assigments for ψ, negative end assignments for ψ) $\models\ \psi$.

Let $S(\psi)$ be the set of assignments s in end positions $(\mathfrak{A}, s, +, \psi)$ reachable in plays where $\mathbb{A}$ follows his strategy. Let $T(\psi)$ be the corresponding set for end positions $(\mathfrak{A}, s, -, \psi)$. Then $S(\psi) \in Q_{\psi}^{\mathfrak{A}}$ and $T(\psi) \in P_{\psi}^{\mathfrak{A}}$, where (Q, P) is the pair of quantifiers that defines the atom ψ. (In other words, $(S(\psi), T(\psi)) \models \psi$.)

Theorem

$\mathfrak{A}, (U, V) \models \varphi$ iff $\mathbb{A}$ has a uniform survival strategy in $G(\mathfrak{A}, U, V, \varphi)$.

Minor quantifiers

What does $\mathfrak{A}, (\{s\}, \emptyset) \models \exists x\, P(x)$ mean in the double team semantics?

It means that $\mathbb{A}$ can *classify* all the elements $a \in Dom(\mathfrak{A})$ according to whether $P(a)$ holds or not, and indeed some elements are in P.

Let T be the trivial quantifier that accepts *any* set. $\mathfrak{A}, (\{s\}, \emptyset) \models Tx\, P(x)$ means that $\mathbb{A}$ can *classify* all the elements $a \in Dom(\mathfrak{A})$ according to whether $P(a)$ holds or not.

These are *constructive statements* that say something more than ordinary generalized quantifiers.

Minor quantifiers (or constructive quantifiers)

To verify $\mathfrak{A} \models_{\mathrm{FO}} \exists x P(x)$, it suffices to find a single witness for $P(x)$; this leads to the definition of the *strict quantifier* $\exists^s$. The generalized quantifier $\exists$ in the double team framework could be characterized as the (epistemically) *total existential quantifier*.

To verify $\mathfrak{A} \models_{\mathrm{FO}} \exists^{<k} x\, P(x)$, it suffices to find a set B whose complement in $\mathfrak{A}$ contains at most $k-1$ elements, and all elements in B falsify $P(x)$. This leads to the definition of *minor counting quantifiers*.

Minor quantifiers

We can add minor quantifiers to the double team framework. The game theoretic semantics also accomodates minor quantifiers in a natural way. Thereby the strict ($\exists^s$) and lax ($\exists^l$) quantifiers are simply minor quantifiers.

Using minor quantifiers and generalized atoms, it is easy to define logic DC^2, that adds counting quantifiers to two-variable dependence logic.

$$\mathrm{D}^2 \;\cup\; \mathrm{FOC}^2 \;\;\subseteq\;\; \mathrm{DC}^2.$$

Theorem

DC^2 *is* NEXPTIME*-complete.*

Proof.

Roughly, use relation symbols to encode teams in order to facilitate a translation into FOC^2. But this is far from the whole story. □

Minor quantifiers

Let Q be a generalized quantifier of the type (1). Let $\mathcal{C}$ be a class of structures (A, B_+, B_-) such that the following conditions hold.

1. $B_+ \subseteq A$ and $B_- \subseteq A$.
2. $B_+ \cap B_- = \emptyset$.
3. For each $(A, B_+, B_-) \in \mathcal{C}$, there exists a pair $(A, H) \in Q$ such that $B_+ \subseteq H$ and $B_- \subseteq A \setminus H$.
4. If $(A, B_+, B_-) \in \mathcal{C}$, there does *not* exist a pair $(A, H) \in \overline{Q}$ such that $B_+ \subseteq H$ and $B_- \subseteq A \setminus H$.
5. For each $(A, H) \in Q$, there exists a tuple $(A, B_-, B_-) \in \mathcal{C}$ such that $B_+ \subseteq H$ and $B_- \subseteq A \setminus H$.
6. $\mathcal{C}$ is closed under isomorphism.

We write $\mathcal{C} \leq Q$. A minor quantifier is a pair $(\mathcal{C}, \mathcal{D})$, where $\mathcal{C} \leq Q$ and $\mathcal{D} \leq \overline{Q}$.