Generalized atoms and quantifiers

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\[ \exists x \ \exists x \ \hat{x} \ \check{x} \]

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We generalize the notion of a Lindström quantifier. The novel minor quantifiers, or constructive quantifiers, are motivated by team semantics as well as independently of team semantics.

We develop a team semantics for a logic with any minor quantifiers and generalized atoms.

We also devise a corresponding game theoretic semantics. The game theoretic semantics is fully symmetric (w.r.t. negation).

We use the resulting semantics by extending two-variable dependence logic with counting quantifiers. We show that the resulting logic DC$^2$ is NEXPTIME-complete.
Double team semantics

**A, (U, V) \models R(x_1, \ldots, x_m) \iff \forall s \in U (A, s \models_{FO} R(x_1, \ldots, x_m)) \text{ and } \forall s \in V (A, s \not\models_{FO} R(x_1, \ldots, x_m)).**

**A, (U, V) \models \neg \varphi \iff A, (V, U) \models \varphi.**

**A, (U, V) \models (\varphi \lor \psi) \iff A, (T, V) \models \varphi \text{ and } A, (S, V) \models \psi \text{ for some } T, S \subseteq U \text{ such that } T \cup S = U.**
Verify that half of the numbers in 6 are odd, i.e., show that
\[
\{0, 1, 2, 3, 4, 5\} \models \text{half of the numbers are odd}.
\]

It suffices to find a subset \(S \subseteq 6\) containing half of the elements in 6 such that

1. each \(x \in S\) is odd,
2. none of the elements \(x\) in \(6 \setminus S\) is odd.
Double team semantics

\[ \mathfrak{A}, (U, V) \models Qx \varphi \]

if and only if there exist functions \( f : U \to Q^\mathfrak{A} \) and \( g : V \to \overline{Q}^\mathfrak{A} \) such that

\[ \mathfrak{A}, (U[x/f] \cup V[x/g], U[x/f'] \cup V[x/g']) \models \varphi. \]

Here \( Q^\mathfrak{A} \) is a generalized quantifier restricted to \( \mathfrak{A} \). For example, \( Q^\mathfrak{A} \) could contain exactly all sets \( B \subseteq A = \text{Dom}(\mathfrak{A}) \) that cover half of the elements of \( \mathfrak{A} \).

\( \overline{Q}^\mathfrak{A} \) would then contain exactly all \( B \subseteq A \) that do not cover half of the elements of \( \text{Dom}(\mathfrak{A}) \).

\[
\begin{align*}
U[x/f] &= \{ s[x/b] \mid s \in U, \ b \in f(s) \}, \\
U[x/f'] &= \{ s[x/b] \mid s \in U, \ b \in A \setminus f(s) \}.
\end{align*}
\]
Generalized atoms

Let \((Q, P)\) be a pair of generalized quantifiers of the type \((i_1, \ldots, i_k)\). The pair \((Q, P)\) defines a generalized atom \(\mathcal{A}\).

Define \(\mathcal{A}, (U, V) = \mathcal{A}(\overline{x}_1, \ldots, \overline{x}_k)\) iff

\[
(\text{Rel}(U, \mathcal{A}, \overline{x}_1), \ldots, \text{Rel}(U, \mathcal{A}, \overline{x}_k)) \in Q^{\mathcal{A}},
\]

and

\[
(\text{Rel}(V, \mathcal{A}, \overline{x}_1), \ldots, \text{Rel}(V, \mathcal{A}, \overline{x}_k)) \in P^{\mathcal{A}}.
\]

\(\text{Rel}(U, \mathcal{A}, \overline{x}_i)\) is the relation induced by the tuple \(\overline{x}_i\) and team \(U\).
Double team semantics

**Theorem**

*For any formula without generalized atoms (but possibly containing generalized quantifiers), \( \mathfrak{A}, (U, V) \models \varphi \) if and only if we have \( \forall s \in U \left( \mathfrak{A}, s \models_{\text{FO}} \varphi \right) \) and \( \forall t \in V \left( \mathfrak{A}, t \not\models_{\text{FO}} \varphi \right) \).*
Positions of game $G(\mathcal{A}, U, V, \varphi)$ are tuples $(\mathcal{A}, s, \#, \psi)$, where $s$ is an assignment, $\# \in \{+,-\}$, and $\psi$ is a subformula of $\varphi$. (Two identical but distinct subformulae are considered different subformulae.)

Two players, *interrogator* $\mathcal{I}$ and *agent* $\mathcal{A}$.

When $\# = +$, the player $\mathcal{A}$ wishes to verify $\psi$. When $\# = -$, the player $\mathcal{A}$ wishes to falsify $\psi$. 

In the beginning of a play of $G(\mathcal{A}, U, V, \varphi)$, $\Pi$ pics a beginning position $(\mathcal{A}, s, +, \varphi)$, where $s \in U$, or $(\mathcal{A}, t, -, \varphi)$, where $t \in V$.

The game is rather similar to the FO semantic game, but the quantifier moves and winning condition differ.

In a position $(\mathcal{A}, s, +, Qx \psi)$, the player $\mathcal{A}$ pics a set $B \in Q^{\mathcal{A}}$. Then $\Pi$ pics an element $b \in B$ or an element $b' \in B \setminus \text{Dom}(\mathcal{A})$. In the first case, the play continues from $(\mathcal{A}, s[x \mapsto b], +, \psi)$, and in the latter case from $(\mathcal{A}, s[x \mapsto b'], -, \psi)$.

(If $Q^{\mathcal{A}} = \emptyset$, the play ends.)
In a position \((A, s, \neg, Q \times \psi)\), the player \(A\) picks a set \(B \in Q^A\). Then \(I\) picks an element \(b \in B\) or an element \(b' \in B \setminus \text{Dom}(A)\). In the first case, the play continues from \((A, s[x \mapsto b], +, \psi)\), and in the latter case from \((A, s[x \mapsto b'], -, \psi)\).

In a position \((A, s, \# , \neg \psi)\), the symbol \(\#\) changes (from + to −, or vice versa). The play continues from \((A, s, \# , \psi)\).
Moves in a position \((\mathcal{A}, s, +, \psi \lor \chi)\) depend on the type of disjunction used. Consider the semantics \((U, V) \models \psi \lor \chi\) iff \((S, V) \models \psi\) and \((T, V) \models \chi\) for \(S, T\) such that \(S \cup T\) and \(S \cap T = \emptyset\).

Then \(\mathbb{A}\) picks one of the positions \((\mathcal{A}, s, +, \chi)\) and \((\mathcal{A}, s, +, \psi)\), and the play continues from there. In a position \((\mathcal{A}, s, -, \psi \lor \chi)\), the player \(\mathbb{I}\) picks one of the positions \((\mathcal{A}, s, -, \chi)\) and \((\mathcal{A}, s, -, \psi)\).
In an end position \((A, s, +, \psi)\), where \(\psi\) is a first-order atom, the player \(A\) survives the play iff \(A, s \models \psi\).

In an end position \((A, s, -, \psi)\), where \(\psi\) is a first-order atom, the player \(A\) survives the play iff \(A, s \not\models \psi\).

In an end position \((A, s, \#, \psi)\), where \(\psi\) is a generalized atom, the player \(A\) survives the play.
A has a *uniform survival strategy* if his strategy guarantees survival, and for each generalized atom \( \psi \) the following holds.

Uniformity condition:

\[(\text{positive end assignments for } \psi, \text{negative end assignments for } \psi) \models \psi.\]

Let \( S(\psi) \) be the set of assignments \( s \) in end positions \((A, s, +, \psi)\) reachable in plays where \( A \) follows his strategy. Let \( T(\psi) \) be the corresponding set for end positions \((A, s, -, \psi)\). Then \( S(\psi) \in Q_\psi \) and \( T(\psi) \in P_\psi \), where \((Q, P)\) is the pair of quantifiers that defines the atom \( \psi \). (In other words, \((S(\psi), T(\psi)) \models \psi.\))
A, (U, V) |= \varphi \text{ iff } A \text{ has a uniform survival strategy in } G(A, U, V, \varphi).
What does $\mathcal{A}, (\{s\}, \emptyset) \models \exists x P(x)$ mean in the double team semantics?

It means that $\mathcal{A}$ can classify all the elements $a \in \text{Dom}(\mathcal{A})$ according to whether $P(a)$ holds or not, and indeed some elements are in $P$.

Let $T$ be the trivial quantifier that accepts any set. $\mathcal{A}, (\{s\}, \emptyset) \models T x P(x)$ means that $\mathcal{A}$ can classify all the elements $a \in \text{Dom}(\mathcal{A})$ according to whether $P(a)$ holds or not.

These are constructive statements that say something more than ordinary generalized quantifiers.
Minor quantifiers (or constructive quantifiers)

To verify $\mathcal{A} \models_{\text{FO}} \exists x P(x)$, it suffices to find a single witness for $P(x)$; this leads to the definition of the strict quantifier $\exists^s$. The generalized quantifier $\exists$ in the double team framework could be characterized as the (epistemically) total existential quantifier.

To verify $\mathcal{A} \models_{\text{FO}} \exists^<k x P(x)$, it suffices to find a set $B$ whose complement in $\mathcal{A}$ contains at most $k - 1$ elements, and all elements in $B$ falsify $P(x)$. This leads to the definition of minor counting quantifiers.
Minor quantifiers

We can add minor quantifiers to the double team framework. The game theoretic semantics also accommodates minor quantifiers in a natural way. Thereby the strict ($\exists^s$) and lax ($\exists^l$) quantifiers are simply minor quantifiers.

Using minor quantifiers and generalized atoms, it is easy to define logic $\text{DC}^2$, that adds counting quantifiers to two-variable dependence logic.

$$\text{D}^2 \cup \text{FOC}^2 \subseteq \text{DC}^2.$$
Theorem

$DC^2$ is NEXPTIME-complete.

Proof.

Roughly, use relation symbols to encode teams in order to facilitate a translation into $FOC^2$. But this is far from the whole story.
Let $Q$ be a generalized quantifier of the type (1). Let $C$ be a class of structures $(A, B_+, B_-)$ such that the following conditions hold.

1. $B_+ \subseteq A$ and $B_- \subseteq A$.
2. $B_+ \cap B_- = \emptyset$.
3. For each $(A, B_+, B_-) \in C$, there exists a pair $(A, H) \in Q$ such that $B_+ \subseteq H$ and $B_- \subseteq A \setminus H$.
4. If $(A, B_+, B_-) \in C$, there does not exist a pair $(A, H) \in \overline{Q}$ such that $B_+ \subseteq H$ and $B_- \subseteq A \setminus H$.
5. For each $(A, H) \in Q$, there exists a tuple $(A, B_-, B_-) \in C$ such that $B_+ \subseteq H$ and $B_- \subseteq A \setminus H$.
6. $C$ is closed under isomorphism.

We write $C \leq Q$. A minor quantifier is a pair $(C, D)$, where $C \leq Q$ and $D \leq \overline{Q}$.