Dependence and Independence in Social Choice Theory

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Competing desiderata for a group decision

1. The voters' preferences should completely determine the group decision.
2. The group decision should depend *in the right way* on the voters' opinions.
3. The voters are free to adopt any preference ordering and the voters' opinions are independent of each other (unless there is good reason to think otherwise).

Competing desiderata for a group decision

1. The voters' preferences should completely determine the group decision. (Dependence)
2. The group decision should depend *in the right way* on the voters' opinions. (Dependence)
3. The voters are free to adopt any preference ordering and the voters' opinions are independent of each other (unless there is good reason to think otherwise). (Independence)

Notation: Candidates, Voters

- N is a finite set of voters (assume that $N = \{1, 2, 3, \ldots, n\}$)

- X is a (typically finite) set of alternatives: e.g., candidates, restaurants, social states, etc.

Notation: Preferences

- A relation on X is a linear order if it is transitive, irreflexive, and complete (hence, acyclic)
- $L(X)$ is the set of all linear orders over the set X
- $O(X)$ is the set of all reflexive, transitive and complete relations over the set X
- Given $R \in O(X)$, let the strict subrelation be $P_R = \{(x, y) \mid x\, R\, y \text{ and } y \not R\, x\}$ and the indifference subrelation be $I_R = \{(x, y) \mid x\, R\, y \text{ and } y\, R\, x\}$

Notation: Profiles

- A **profile** for the set of voters N is a sequence of (linear) orders over X, denoted $\mathbf{R} = (R_1, \ldots, R_n)$.
- $L(X)^n$ is the set of all **profiles** for n voters (similarly for $O(X)^n$)
- For a profile $\mathbf{R} = (R_1, \ldots, R_n) \in O(X)^n$, let $\mathbf{N_R}(A\ P\ B) = \{i \mid A\ P_i\ B\}$ be the set of voters that rank A above B (similarly for $\mathbf{N_R}(A\ I\ B)$ and $\mathbf{N_R}(B\ P\ A)$)

Group Decision Making Methods

$$F : \mathcal{D} \to \mathcal{R}$$

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Comments

- $\mathcal{D}$ is the *domain* of the function: the set of possible "election scenarios" (i.e., $\mathcal{D} \subseteq L(X)^n$, $\mathcal{D} \subseteq O(X)^n$, $\mathcal{D} \subseteq U(X)^n$, where $U(X)$ is the set of utility functions on X)

- The group decision is *completely determined* by the voters' opinions: every profile $\mathbf{R} \in \mathcal{D}$ is associated with exactly one "group decision".

Group Decision Making Methods

$$F : \mathcal{D} \rightarrow \mathcal{R}$$

Variants

- Social Welfare Functions: $\mathcal{R} = L(X)$ or $\mathcal{R} = O(X)$.
- Social Choice Function: $\mathcal{R} = \wp(X) - \emptyset$, where $\wp(X)$ is the set of all subsets of X.

Group Decision Making Methods

$$F : \mathcal{D} \rightarrow \mathcal{R}$$

Variants

- $\mathcal{D} = J(X)^n$ and $\mathcal{R} = J(X)$, where $J(X)$ is the set of complete, consistent subsets of a set X of propositional formulas
- $\mathcal{D} = U(X)^n$ and $\mathcal{R} = U(X)$ where $U(X)$ is the set of utility functions
- $\mathcal{D} = \Delta(X)^n$ and $\mathcal{R} = \Delta(X)$ where $\Delta(X)$ is the set of probability measures on X ($p_i(A)$ is the probability that i would choose A if i could act as a dictator.)

The setup (from Jouko's various talks)

The set of variables is $V = \{x_1, x_2, \ldots, x_n\} \cup \{y\}$ (each x_i is a voter and y is the social outcome)

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A team S is a set of possible election scenarios

The setup (from Jouko's various talks)

(Atomic) formulas describe the voters' preferences and the group decision.

- $P_{AB}(x_i)$ is true if $s(x_i)$ ranks A strictly above B,
- $R_{AB}(x_i)$ is true if A is weakly preferred to B (i.e., $A\ s(x_i)\ B$)
- $P_{AB}(y)$ is true when the group ranks A strictly above B
- $C_A(y)$ means that A was chosen by the group,
- $\overline{C_A}(y)$ means that the group does not choose A
- $\ldots$

An underlying theory describing (individual and group) rationality assumptions,
E.g., transitivity: $P_{AB}(x) \wedge P_{BC}(x) \supset P_{AC}(x)$
Resolute rules, $C_A(y) \supset \overline{C_B}(y)$,
$P_{AB}(x) \supset (R_{AB}(x) \wedge \neg R_{AB}(x))$, etc.

Desiderata 1

The voters' preferences should completely determine the group decision.

$=(x_1, x_2, \ldots, x_n, y)$

Desiderata 2

The group decision should depend *in the right way* on the voters' opinions.

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- Single-profile conditions
- Multi-profile conditions
- Variable domain conditions

Single-Profile Conditions

Condorcet: Elect the Condorcet winner whenever it exists.

A Condorcet candidate in a profile **R** is a candidate A such that $|\mathbf{N_R}(A\ P\ B)| > |\mathbf{N_R}(B\ P\ A)|$ for all other candidates $B \in X$

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Pareto: Never elect a candidate that is dominated.

Weak Pareto: for all profiles $\mathbf{R}$, if for all $i \in N$, $A\ P_i\ B$, then $A\ P_{F(\mathbf{R})}\ B$ (recall that P_i is the strict subrelation of $\mathbf{R}_i$)

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$(\bigwedge_{i \in N} P_{AB}(x_i)) \supset P_{AB}(y)$

(*needs to be stated for every pair* A, B *of candidates*)

Multi-Profile Conditions

IIA: The group's ranking of A and B should only depend on the voter's rankings of A and B

for all profiles $\mathbf{R}, \mathbf{R}'$ if for all $i \in N$, $\mathbf{R}_i|_{\{A,B\}} = \mathbf{R}'_i|_{\{A,B\}}$, then $F(\mathbf{R})|_{\{A,B\}} = F(\mathbf{R}')|_{\{A,B\}}$

$=(R_{AB}(x_1), \ldots, R_{AB}(x_n), R_{AB}(y))$

Multi-Profile Conditions, continued

Anonymity: The outcome does not depend on the names of the voters.

If π is a permutation of the voters, for all profiles $\mathbf{R} = (R_1, \ldots, R_n)$, $\mathbf{R}'$, if $\mathbf{R}' = (R_{\pi(1)}, \ldots, R_{\pi(n)})$, then $F(\mathbf{R}) = F(\mathbf{R}')$. (not the same as $=(x_{\pi(1)}, \ldots, x_{\pi(n)}, y)$)

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Neutrality: The outcome does not depend on the names of the candidates.

Monotonicity: More support should never hurt a candidate.

Early Criticism of Multi-Profile Conditions

"If tastes change, we may expect a new ordering of all the conceivable states; but we do not require that the difference between the new and the old ordering should bear any particular relation to the changes of taste which have occurred. We have, so to speak, a new world and a new order, and we do not demand correspondence between the change in the world and the change in the order" (pg. 423-424)

I. Lilttle. *Social Choice and Individual Values*. Journal of Political Economy, 60:5, pgs. 422 - 432, 1952.

Variable Domain Conditions

Participation: It should never be in a voter's best interests not to vote.

Multiple-Districts: If a candidate wins in each district, then that candidate should also win when the districts are merged.

Variable Population Model

Let $\mathbb{N}$ be the set of "potential" voters.

Let $\mathcal{V} = \{V \mid V \subseteq \mathbb{N}$ and V is finite$\}$ be the set of all voting blocks.

For $V \in \mathcal{V}$, a **profile** for V is a function $\pi : V \to \mathcal{P}$ where $\mathcal{P}$ is $O(X)$, $L(X)$ or some set $\mathcal{B}$ of "ballots".

Let $\Pi^{\mathcal{P}}$ be the set of all profiles based on $\mathcal{P}$.

Variable Population Model

Two profiles $\pi : V \to \mathcal{P}$ and $\pi' : V' \to \mathcal{P}$ are *disjoint* if $V \cap V' = \emptyset$

If $\pi : V \to \mathcal{P}$ and $\pi' : V' \to \mathcal{P}$ are *disjoint*, then $(\pi + \pi') : (V \cup V') \to \mathcal{P}$ is the profile where for all $i \in V \cup V'$,

$$(\pi + \pi')(i) = \begin{cases} \pi(i) & \text{if } i \in V \\ \pi'(i) & \text{if } i \in V' \end{cases}$$

Consistency: If $F(\pi) \cap F(\pi') \neq \emptyset$, then $F(\pi + \pi') = F(\pi) \cap F(\pi')$

What are the relationships between these principles? Is there a procedure that satisfies *all* of them?

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A few observations:

- Condorcet winners may not exist.
- No positional scoring method satisfies the Condorcet Principle.
- The Condorcet and Participation principles cannot be jointly satisfied (Moulin's Theorem).

Desiderata 3

The voters are free to adopt any preference ordering and the voters' opinions are independent of each other (unless there is good reason to think otherwise).

Domain Conditions

Universal Domain: The domain of the social welfare (choice) function is $\mathcal{D} = L(X)^n$ (or $O(X)^n$)

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Epistemic Rationale: "If we do not wish to require any prior knowledge of the tastes of individuals before specifying our social welfare function, that function will have to be defined for every logically possible set of individual orderings." (Arrow, 1963, pg. 24)

Domain Conditions

No Restrictions:

$S \models \text{all}(x_i)$ if for all $R \in L(X)$, there is an $s \in S$ such that $s(x_i) = R$

$S \models \text{triple}(x_i)$ if for all $P \in L(\{A, B, C\})$ there is a $R' \in O(X)$ and $s \in X$ such that $R'|_{\{A,B,C\}} = P$ and $s(x_i) = R'$

$\bigwedge_{i=1}^{n} all(x_i)$

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$\bigwedge_{i=1}^{n} all(x_i)$

Independence:

$\{x_j \mid j \neq i\} \perp x_i$

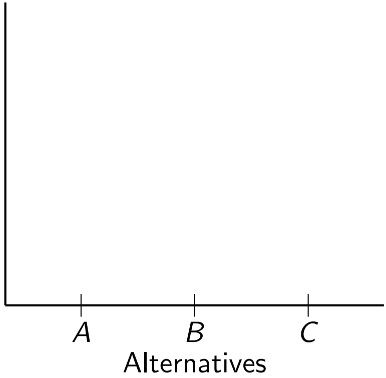
Domain Restrictions

- Single-Peaked preferences
- Sen's Value Restriction
- Assumptions about the distribution of preferences

W. Gaertner. *Domain Conditions in Social Choice Theory*. Cambridge University Press, 2001.

1	1	1
A	B	C
B	C	A
C	A	B

1	1	1
A	B	C
B	C	A
C	A	B



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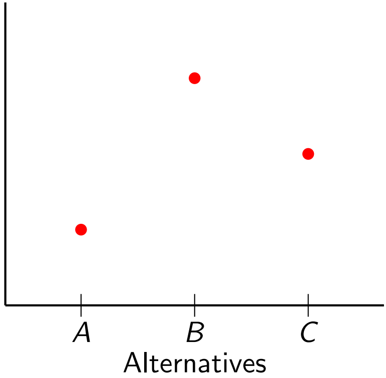

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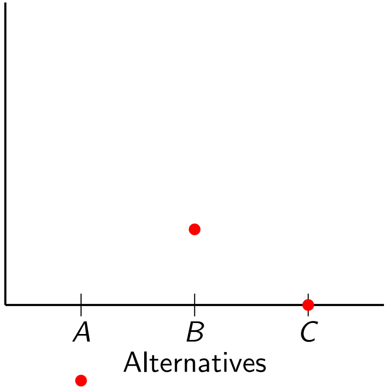

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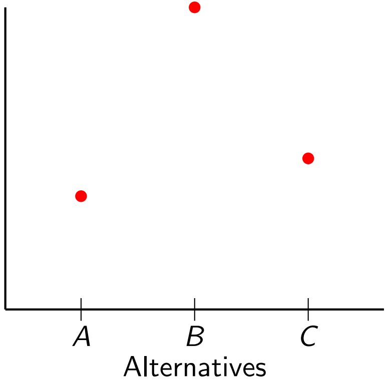

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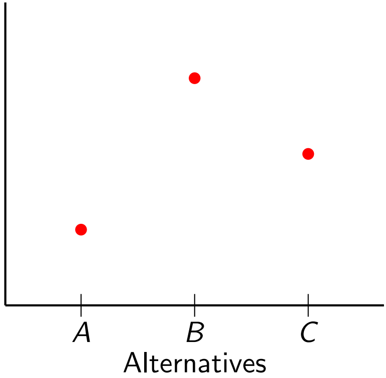

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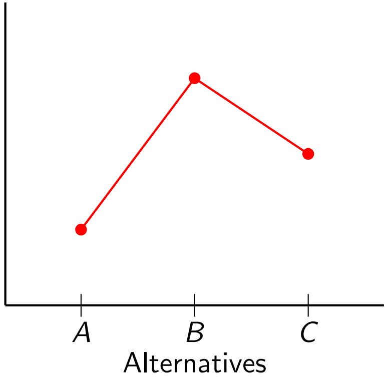

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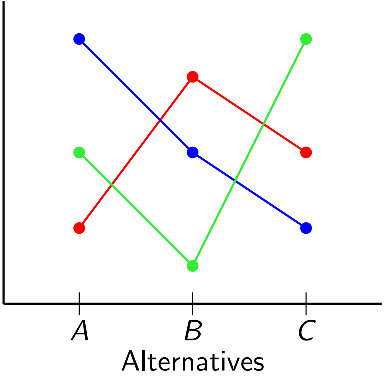

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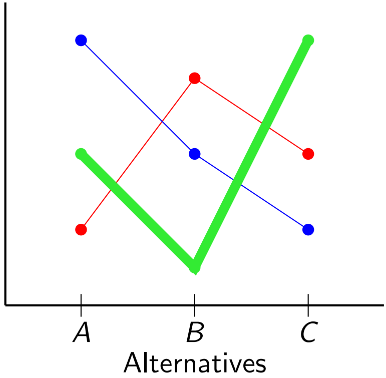
C	A	B



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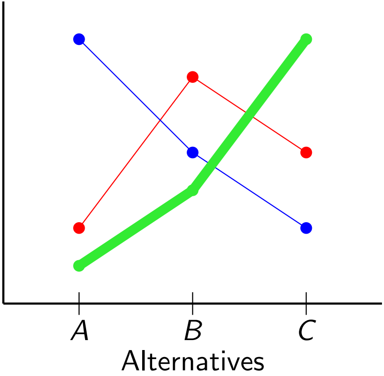
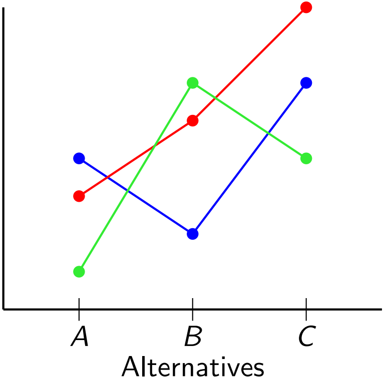
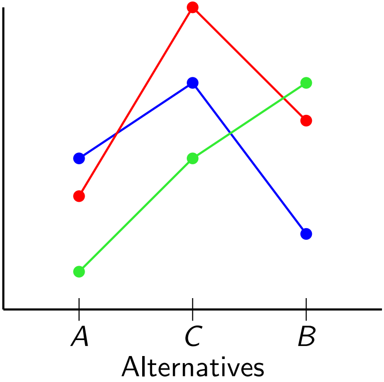


1	1	1
C	*C*	*B*
A	*B*	*C*
B	*A*	*A*



1	1	1
C	C	B
A	B	C
B	A	A



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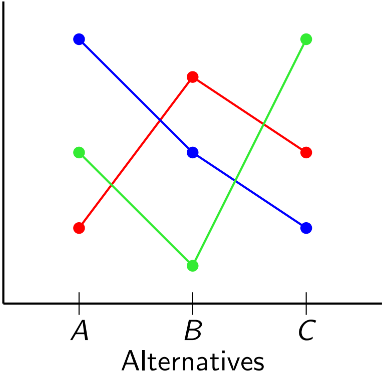

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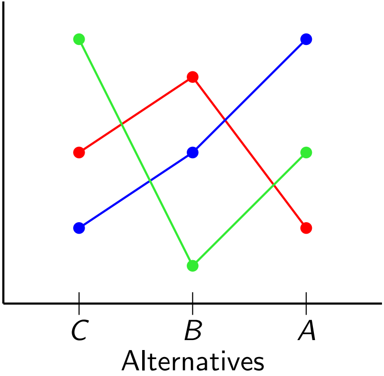

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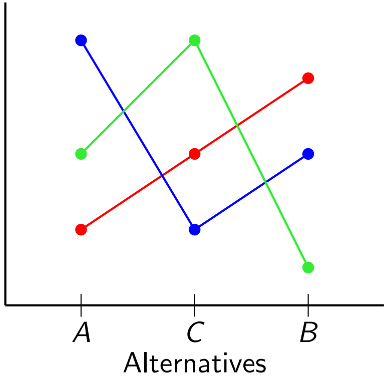

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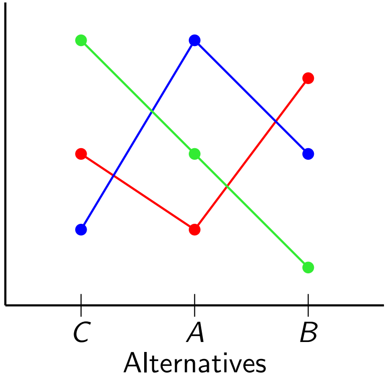

D. Black. *On the rationale of group decision-making*. Journal of Political Economy, 56:1, pgs. 23 - 34, 1948.

Single-Peakedness: the preferences of group members are said to be single-peaked if the alternatives under consideration can be represented as points on a line and each of the utility functions representing preferences over these alternatives has a maximum at some point on the line and slopes away from this maximum on either side.

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Theorem. If there is an odd number of voters that display single-peaked preferences, then a Condorcet winner exists.

D. Miller. *Deliberative Democracy and Social Choice*. Political Studies, 40, pgs. 54 - 67, 1992.

C. List, R. Luskin, J. Fishkin and I. McLean. *Deliberation, Single-Peakedness, and the Possibility of Meaningful Democracy: Evidence from Deliberative Polls*. Journal of Politics, 75(1), pgs. 80 - 95, 2013.

Sen's Value Restriciton

A. Sen. *A Possibility Theorem on Majority Decisions*. Econometrica 34, 1966, pgs. 491 - 499.

Sen's Theorem

Assume n voters (n is odd).

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Triplewise value-restriction: For every triple of distinct candidates A, B, C there exists an $x_i \in \{A, B, C\}$ and $r \in \{1, 2, 3\}$ such that no voter ranks x_i has her rth preference among A, B, C.

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Theorem (Sen, 1966). For every profile satisfying triplewise value-restriction, pairwise majority voting generates a transitive group preference ordering.

Restrict the *distribution* of preferences

M. Regenwetter, B. Grofman, A.A.J. Marley and I. Tsetlin. *Behavioral Social Choice*. Cambridge University Press, 2006.

Let $\mathbb{P}$ be a probability on $L(X)$.

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The **net probability** induced by $\mathbb{P}$ is: $N\!P(R) = \mathbb{P}(R) - \mathbb{P}(R^{-1})$, where $R \in L(X)$ and R^{-1} in the inverse of R ($A\ R^{-1}\ B$ iff $B\ R\ A$).

$N\!P_{ABC} = \mathbb{P}_{ABC} - \mathbb{P}_{CBA}$

Fix three candidates $\{A, B, C\}$

$\mathcal{N}\mathcal{P}$ satisfies $NW(C)$ iff $\mathcal{N}\mathcal{P}_{ABC} \leq 0$ and $\mathcal{N}\mathcal{P}_{BAC} \leq 0$

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$\mathcal{NP}$ is marginally value restricted for the triple $\{Y, Z, W\}$ iff there is an element $C \in \{Y, Z, W\}$ such that $\mathcal{NP}$ satisfies $NW(c)$, $NB(c)$ or $NB(c)$. Net value restriction holds on X if marginal net value restrictions holds on each triple.

Consider a probability $\mathbb{P}$ on $L(X)$. A weak majority preference relation $\succeq$ and a strict majority preference relation $\succ$ are defined as follows:

$$A \succeq B \text{ iff } \mathbb{P}_{AB} \geq \mathbb{P}_{BA}$$

$$A \succ B \text{ iff } \mathbb{P}_{AB} > \mathbb{P}_{BA}$$

Theorem (Regenwetter et al.). The weak majority preference relations is transitive iff for each triple $\{A, B, C\} \subseteq X$ at least one of the following two conditions holds:

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2. There is a $R_0 \in \{ABC, ACB, BAC, BCA, CAB, CBA\}$ such that R_0 has marginal net preference majority.

Theorem (Regenwetter et al.). The weak majority preference relations is transitive iff for each triple $\{A, B, C\} \subseteq X$ at least one of the following two conditions holds:

1. $N\!P$ is marginally value restricted on $\{A, B, C\}$ and, in addition, if at least one net preference is nonzero then the following implication is true $N\!P_{ABC} = 0 \Rightarrow N\!P_{BAC} \neq N\!P_{ACB}$ (with possible relabelings).
2. There is a $R_0 \in \{ABC, ACB, BAC, BCA, CAB, CBA\}$ such that R_0 has marginal net preference majority.

We say CDE has net preference majority provided:

$$N\!P_{CDE} > \sum_{R' \in \{CED, DEC, DCE, ECD, EDC\}, N\!P(R') > 0} N\!P_{R'}$$

Decisiveness

A voter i is decisive for A over B provided for all profiles $\mathbf{R}$, if $A\ P_i\ B$, then $A\ P_{F(\mathbf{R})}\ B$.

$=(P_{AB}(x_i), P_{AB}(y))$

(Here, note that we have $\neg P_{AB}(x_i) \leftrightarrow P_{BA}(x_i)$)

$$F : L(X)^n \to (\wp(X) - \emptyset)$$

$F : L(X)^n \rightarrow (\wp(X) - \emptyset)$

Pareto: For all profiles $\mathbf{R} \in L(X)^n$ and alternatives A, B, if $A\ R_i\ B$ for all $i \in N$, then $B \notin F(\mathbf{R})$.

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Liberalism: For all voters $i \in N$, there exists two alternatives A_i and B_i such that for all profiles $\mathbf{R} \in L(X)^n$, if $A_i\ R_i\ B_i$, then $B \notin F(\mathbf{R})$. That is, i is **decisive** over A_i and B_i.

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Minimal Liberalism: There are two distinct voters i and j such that there are alternatives A_i, B_i, A_j, and B_j such that i is decisive over A_i and B_i and j is decisive over A_j and B_j.

Sen's Impossibility Theorem. Suppose that X contains at least three elements. No social choice function $F : L(X)^n \rightarrow (\wp(X) - \emptyset)$ satisfies (universal domain) and both minimal liberalism and the Pareto condition.

A. Sen. *The Impossibility of a Paretian Liberal*. Journal of Political Economy, 78:1, pp. 152 - 157, 1970.

Suppose that X contains at least three elements and there are elements A, B, C and D such that

1. Voter 1 is decisive over A and B: for any profile $\mathbf{R} \in L(X)^n$, if $A \; R_1 \; B$, then $B \notin F(\mathbf{R})$
2. Voter 2 is decisive over C and D: for any profile $\mathbf{R} \in L(X)^n$, if $C \; R_2 \; D$, then $D \notin F(\mathbf{R})$

Two cases: 1. $B \neq C$ and 2. $B = C$.

Suppose that $X = \{A, B, C, D\}$ and

- Voter 1 is decisive over the pair A, B
- Voter 2 is decisive over the pair C, D

1	2
D	B
A	C
B	D
C	A

1	2
D	B
A	C
B	D
C	A

Voter 1 is decisive for A, B implies $B \not\in F(\mathbf{R})$

1	2
D	B
A	C
B	D
C	A

Voter 1 is decisive for A, B implies $B \notin F(\mathbf{R})$

Voter 2 is decisive for C, D implies $D \notin F(\mathbf{R})$

1	2
D	B
A	C
B	D
C	A

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Voter 2 is decisive for C, D implies $D \notin F(\mathbf{R})$

Pareto implies $A \notin F(\mathbf{R})$

1	2
D	B
A	C
B	D
C	A

Voter 1 is decisive for A, B implies $B \notin F(\mathbf{R})$

Voter 2 is decisive for C, D implies $D \notin F(\mathbf{R})$

Pareto implies $A \notin F(\mathbf{R})$

Pareto implies $C \notin F(\mathbf{R})$

Suppose that $X = \{A, B, C\}$ and

- Voter 1 is decisive over the pair A, B
- Voter 2 is decisive over the pair B, C
- Voter 1's preference $R_1 \in L(X)$ is $C \; R_1 \; A \; R_1 \; B$
- Voter 2's preference $R_2 \in L(X)$ is $B \; R_2 \; C \; R_2 \; A$

1	2
C	B
A	C
B	A

1	2
C	B
A	C
B	A

Voter 1 is decisive for A, B implies $B \not\in F(\mathbf{R})$

1	2
C	B
A	C
B	A

Voter 1 is decisive for A, B implies $B \notin F(\mathbf{R})$

Voter 2 is decisive for B, C implies $C \notin F(\mathbf{R})$

1	2
C	B
A	C
B	A

Voter 1 is decisive for A, B implies $B \notin F(\mathbf{R})$

Voter 2 is decisive for B, C implies $C \notin F(\mathbf{R})$

Pareto implies $A \notin F(\mathbf{R})$

“What is the moral?

“What is the moral? It is that in a very basic sense liberal values conflict with the Pareto principle. If someone takes the Pareto principle seriously, as economists seem to do, then he has to face problems of consistency in cherishing liberal values, even very mild ones....

"What is the moral? It is that in a very basic sense liberal values conflict with the Pareto principle. If someone takes the Pareto principle seriously, as economists seem to do, then he has to face problems of consistency in cherishing liberal values, even very mild ones.... While the Pareto criterion has been thought to be an expression of individual liberty, it appears that in choices involving more than two alternatives it can have consequences that are, in fact, deeply illiberal." (pg. 157)

A. Sen. *The Impossibility of a Paretian Liberal*. Journal of Political Economy, 78:1, pp. 152 - 157, 1970.

What's the moral?

What's the moral?

- $\text{all}(x_1) \wedge \text{all}(x_2)$
- $x_1 \perp x_2$
- $(P_{YZ}(x_1) \wedge P_{YZ}(x_2)) \supset P_{YZ}(y)$, for Y, Z distinct elements of $\{A, B, C, D\}$
- $=\!(x_1, x_2, y)$
- $=\!(P_{AB}(x_1), P_{AB}(y))$
- $=\!(P_{CD}(x_2), P_{CD}(y))$

are inconsistent.

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Characterizing Majority Rule

When there are only **two** candidates A and B, then all voting methods give the same results

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When there are only two options, can we argue that majority rule is the "best" procedure?

K. May. *A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision*. Econometrica, Vol. 20 (1952).

May's Theorem: Details

Let $N = \{1, 2, 3, \ldots, n\}$ be the set of n voters and $X = \{A, B\}$ the set of candidates.

Social Welfare Function: $F : O(X)^n \rightarrow O(X)$, where $O(X)$ is the set of orderings over X
(*there are only three possibilities: $A\ P\ B$, $A\ I\ B$, or $B\ P\ A$*)

$$F_{Maj}(\mathbf{R}) = \begin{cases} A\ P\ B & \text{if } |\mathbf{N}_{\mathbf{R}}(A\ P\ B)| > |\mathbf{N}_{\mathbf{R}}(B\ P\ A)| \\ A\ I\ B & \text{if } |\mathbf{N}_{\mathbf{R}}(A\ P\ B)| = |\mathbf{N}_{\mathbf{R}}(B\ P\ A)| \\ B\ P\ A & \text{if } |\mathbf{N}_{\mathbf{R}}(B\ P\ A)| > |\mathbf{N}_{\mathbf{R}}(A\ P\ B)| \end{cases}$$

May's Theorem: Details

Let $N = \{1, 2, 3, \ldots, n\}$ be the set of n voters and $X = \{A, B\}$ the set of candidates.

Social Welfare Function: $F : \{1, 0, -1\}^n \rightarrow \{1, 0, -1\}$,

where 1 means $A\ P\ B$, 0 means $A\ I\ B$, and -1 means $B\ P\ A$

$$F_{Maj}(\mathbf{v}) = \begin{cases} 1 & \text{if } |\mathbf{N_R}(1)| > |\mathbf{N_R}(-1)| \\ 0 & \text{if } |\mathbf{N_R}(1)| = |\mathbf{N_R}(-1)| \\ -1 & \text{if } |\mathbf{N_R}(-1)| > |\mathbf{N_R}(1)| \end{cases}$$

May's Theorem: Details

- **Unanimity**: unanimously supported alternatives must be the social outcome.

- **Anonymity**: all voters should be treated equally.

- **Neutrality**: all candidates should be treated equally.

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  If $\mathbf{v} = (v_1, \ldots, v_n)$ with for all $i \in N$, $v_i = x$ then $F(\mathbf{v}) = x$ (for $x \in \{1, 0, -1\}$).

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- **Anonymity**: all voters should be treated equally.

  $F(v_1, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where $v_i \in \{1, 0, -1\}$ and π is a permutation of the voters.

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- **Neutrality**: all candidates should be treated equally.

  $F(-\mathbf{v}) = -F(\mathbf{v})$ where $-\mathbf{v} = (-v_1, \ldots, -v_n)$.

May's Theorem: Details

- **Positive Responsiveness** (Monotonicity): unidirectional shift in the voters' opinions should help the alternative toward which this shift occurs

If $F(\mathbf{v}) = 0$ or $F(\mathbf{v}) = 1$ and $\mathbf{v} \prec \mathbf{v}'$, then $F(\mathbf{v}') = 1$ where $\mathbf{v} \prec \mathbf{v}'$ means for all $i \in N$ $v_i \leq v_i'$ and there is some $i \in N$ with $v_i < v_i'$.

May's Theorem: Details

May's Theorem (1952) A social decision method F satisfies unanimity, neutrality, anonymity and positive responsiveness iff F is majority rule.

Proof Idea

If $(1, 0, -1)$ is assigned 1 or -1 then

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- ✓ Neutrality implies $(1, 0, -1)$ is assigned -1 or 1
  Contradiction.

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- ✓ Positive Responsiveness implies $(1, 1, -1)$ is assigned 1
  Contradiction.

Other characterizations

G. Asan and R. Sanver. *Another Characterization of the Majority Rule*. Economics Letters, 75 (3), 409-413, 2002.

E. Maskin. *Majority rule, social welfare functions and game forms*. in *Choice, Welfare and Development*, The Clarendon Press, pgs. 100 - 109, 1995.

G. Woeginger. *A new characterization of the majority rule*. Economic Letters, 81, pgs. 89 - 94, 2003.

May's Theorem in Dependence Logic

Let $D = \{-1, 0, 1\}$

$V = \{x_1, x_2, \ldots, x_n, y\}$

Profiles are substitutions: $s : V \rightarrow D$

An election scenario is a set of profiles (i.e., a team).

Let $T(x)$ mean A and B are tied for x, $A(x)$ mean x ranks A above B and $B(x)$ mean x ranks B above A.

Function: $=(x_1, \ldots, x_n, y)$

Unanimity: The conjunction of

- $(\bigwedge_{i=1}^{n} x_i = 0) \supset y = 0$
- $(\bigwedge_{i=1}^{n} x_i = 1) \supset y = 1$
- $(\bigwedge_{i=1}^{n} x_i = -1) \supset y = -1$

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Neutrality: For all $s, s' \in X$, if $s'(x_i) = -s(x_i)$ for all $i = 1, \ldots, n$, then $s'(y) = -s(y)$

Function: $=\!(x_1, \ldots, x_n, y)$

Unanimity: The conjunction of

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Neutrality: For all $s, s' \in X$, if $s'(x_i) = -s(x_i)$ for all $i = 1, \ldots, n$, then $s'(y) = -s(y)$

Positive Responsiveness: For all $s, s' \in X$, if $\bigwedge_{i=1}^{n} s(x_i) \leq s'(x_i)$ and $\bigvee_{i=1}^{n} s(x_i) < s(x_i)$, then $(s(y) = 0$ or $s(y) = 1)$ implies $s'(y) = 1$.

Neutrality and Positive Responsiveness are generalized version of dependency conditions:

Neutrality and Positive Responsiveness are generalized version of dependency conditions:

Suppose that R, R' are relations on the domain M,

$\mathcal{M}, X \models [R, R'](x, y)$ iff for all $s, s' \in X$, if $R(s(x), s'(x))$ then $R(s(y), s'(y))$

$[=, =](x, y)$ is $=(x, y)$

$R_{Neut}(z_1, \ldots, z_n, z'_1, \ldots, z'_n)$ is
$\bigwedge_{i=1}^{n}((T(z_i) \wedge T(z'_i)) \vee (A(z_i) \wedge B(z'_i)) \vee (B(z_i) \wedge A(z'_i))$.

$R'_{Neut}(z, z')$ is $(T(z) \wedge T(z')) \vee (A(z) \wedge B(z')) \vee (B(z) \wedge A(z'))$

$R_{Neut}(z_1, \ldots, z_n, z'_1, \ldots, z'_n)$ is $\bigwedge_{i=1}^{n}((T(z_i) \wedge T(z'_i)) \vee (A(z_i) \wedge B(z'_i)) \vee (B(z_i) \wedge A(z'_i))$.

$R'_{Neut}(z, z')$ is $(T(z) \wedge T(z')) \vee (A(z) \wedge B(z')) \vee (B(z) \wedge A(z'))$

$R_{Mon}(z_1, z_2, \ldots, z_n, z'_1, z'_2, \ldots, z'_n)$ is $\bigwedge_{i=1}^{n} z_i \leq z'_i \wedge \bigvee_{i=1}^{n}(z_i < z'_i)$

$R'_{Mon}(z)$ is $(T(z) \vee A(z)) \supset A(z)$

Arrow's Theorem

K. Arrow. *Social Choice and Individual Values*. John Wiley & Sons, 1951.

Arrovian Dictator

A voter $d \in N$ is a **dictator** if society strictly prefers A over B *whenever* d strictly prefers A over B.

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There is a $d \in N$ such that for each profile $\mathbf{R} = (R_1, \ldots, R_d, \ldots, R_n)$, if $A\ P_d\ B$, then $A\ P_{F(\mathbf{R})}\ B$

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(Moreau's Zelig example)

M. Morreau. *Arrow's Theorem*. Stanford Encyclopedia of Philosophy, forthcoming, 2014.

Arrow's Theorem

- There are at least three candidates and finitely many voters
- $\bigwedge_i \text{all}(x_i)$
- $\bigwedge_i(\{x_j \mid j \neq i\} \perp x_i)$
- $(\bigwedge_i P_{AB}(x_i)) \supset P_{AB}(y)$ (for all pairs A, B)
- $=\!(R_{AB}(x_1), \ldots, R_{AB}(x_n), R_{AB}(y))$ (for all pairs A, B)
- There exists a d such that $=\!(P_{AB}(x_d), P_{AB}(y))$ for all A, B

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies universal domain, independence of irrelevant alternatives and unanimity is a dictatorship.

Arrow's Theorem

D. Campbell and J. Kelly. *Impossibility Theorems in the Arrovian Framework*. Handbook of Social Choice and Welfare Volume 1, pgs. 35 - 94, 2002.

J. Geanakoplos. *Three Brief Proofs of Arrow's Impossibility Theorem*. Economic Theory, **26**, 2005.

P. Suppes. *The pre-history of Kenneth Arrow's social choice and individual values*. Social Choice and Welfare, 25, pgs. 319 - 326, 2005.

Weakening IIA

Given a profile and a set of candidates $S \subseteq X$, let $\mathbf{R}|_S$ denote the restriction of the profile to candidates in S.

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Binary Independence: For all profiles $\mathbf{R}, \mathbf{R}'$ and candidates $A, B \in X$:

$$\text{If } \mathbf{R}|_{\{A,B\}} = \mathbf{R}'|_{\{A,B\}}, \text{ then } F(\mathbf{R})|_{\{A,B\}} = F(\mathbf{R}')|_{\{A,B\}}$$

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m-**Ary Independence**: For all profiles $\mathbf{R}, \mathbf{R}'$ and for all $S \subseteq X$ with $|S| = m$:

$$\text{If } \mathbf{R}|_S = \mathbf{R}'|_S, \text{ then } F(\mathbf{R})|_S = F(\mathbf{R}')|_S$$

Weakening IIA

Theorem. (Blau) Suppose that $m = 2, \ldots, |X| - 1$. If a social welfare function F satisfies m-ary independence, then it also satisfies binary independence.

J. Blau. *Arrow's theorem with weak independence*. Economica, 38, pgs. 413 - 420, 1971.

S. Cato. *Independence of Irrelevant Alternatives Revisited*. Theory and Decision, 2013.

Let $\mathcal{S} \subseteq \wp(X)$. F is $\mathcal{S}$-**independent** if for all profiles $\mathbf{R}, \mathbf{R}'$, and all $S \in \mathcal{S}$,

$$\text{if } \mathbf{R}|S = \mathbf{R}'|S, \text{ then } F(\mathbf{R})|S = F(\mathbf{R}')|S$$

$\mathcal{S} \subseteq \wp(X)$ is **connected** provided for all $x, y \in X$ there is a finite set $S^1, \ldots, S^k \in \mathcal{S}$ such that

$$\{x, y\} = \bigcap_{j \in \{1,\ldots,k\}} S^j$$

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Theorem (Sato). (i) Suppose that $\mathcal{S} \subseteq \wp(X)$ is connected. If a collective choice rule F satisfies $\mathcal{S}$-independence, then it also satisfies binary independence.

(ii) Suppose that $\mathcal{S} \subseteq \wp(X)$ is not connected. Then, there exists a social welfare function F that satisfies $\mathcal{S}$-independence and weak Pareto but does not satisfy binary independence.

Arrow's Theorem

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies universal domain, independence of irrelevant alternatives and unanimity is a dictatorship.

Weakening Unanimity

$F : \mathcal{D} \to O(X)$

Dictatorial: there is a $d \in N$ such that for all $A, B \in X$ and all profiles $\mathbf{R}$:
if $A\ P_d\ B$, then $A\ P_{F(\mathbf{R})}\ B$

Inversely Dictatorial: there is a $d \in N$ such that for all $A, B \in X$ and all profiles $\mathbf{R}$: if $A\ P_d\ B$, then $B\ P_{F(\mathbf{R})}\ A$

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Null: For all $A, B \in X$ and for all $\mathbf{R} \in \mathcal{D}$: $A\ I_{F(\mathbf{R})}\ B$

Weakening Unanimity

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Null: For all $A, B \in X$ and for all $\mathbf{R} \in \mathcal{D}$: $A\ I_{F(\mathbf{R})}\ B$

Non-Imposition: For all $A, B \in X$, there is a $\mathbf{R} \in \mathcal{D}$ such that $A\ F(\mathbf{R})\ B$

Weakening Unanimity

Theorem (Wilson) Suppose that N is a finite set. If a social welfare function satisfies universal domain, independence of irrelevant alternatives and non-imposition, then it is either null, dictatorial or inversely dictatorial.

R. Wilson. *Social Choice Theory without the Pareto principle*. Journal of Economic Theory, 5, pgs. 478 - 486, 1972.

Y. Murakami. *Logic and Social Choice*. Routledge, 1968.

S. Cato. *Social choice without the Pareto principle: A comprehensive analysis*. Social Choice and Welfare, 39, pgs. 869 - 889, 2012.

Arrow's Theorem

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies universal domain, independence of irrelevant alternatives and unanimity is a dictatorship.

Social Choice Functions

$F : \mathcal{D} \to \wp(X) - \emptyset$

Resolute: For all profiles $\mathbf{R} \in \mathcal{D}, |F(\mathbf{R})| = 1$

Non-Imposed: For all candidates $A \in X$, there is a $\mathbf{R} \in \mathcal{D}$ such that $F(\mathbf{R}) = \{A\}$.

Monotonicity: For all profiles $\mathbf{R}$ and $\mathbf{R}'$, if $A \in F(\mathbf{R})$ and for all $i \in N$, $\mathbf{N}_{\mathbf{R}}(A\ P_i\ B) \subseteq \mathbf{N}_{\mathbf{R}'}(A\ P_i'\ B)$ for all $B \in X - \{A\}$, then $A \in F(\mathbf{R}')$.

Dictator: A voter d is a dictator if for all $\mathbf{R} \in \mathcal{D}$, $F(\mathbf{R}) = \{A\}$, where A is d's top choice.

Social Choice Functions

Muller-Satterthwaite Theorem. Suppose that there are more than three alternatives and finitely many voters. Every resolute social choice function $F : L(X)^n \to X$ that is monotonic and non-imposed is a dictatorship.

E. Muller and M.A. Satterthwaite. *The Equivalence of Strong Positive Association and Strategy-Proofness*. Journal of Economic Theory, 14(2), pgs. 412 - 418, 1977.

Arrow's Theorem

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies universal domain, independence of irrelevant alternatives and unanimity is a dictatorship.

Conclusions

- Many generalizations and variants of Arrow's Theorem (e.g., infinite voters), many characterization results
- A number of different logics have been used to formalize various aspects of Arrow's Theorem and related impossibility results
- Dependence logic and social choice: Social choice may suggest new dependence atoms. DL is good match for Social Choice: it focuses on reasoning about dependence (i.e., IIA) and independence (i.e., freedom of choice).