(In)dependence Logic in Model Theory

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(In)dependence in Model Theory

In model theory the concepts of dependence and independence are of crucial importance, it is indeed always in function of an independence calculus that a **classification theory** for a class of classes of structures is developed.

Three are the main frameworks in which (in)dependence has been studied:

- pregeometries;
- first-order theories;
- abstract elementary classes (AECs).

Classification Theory

Pregeometries		
Vector spaces	Alg. closed fields	Graphs

Forking Indep.		
ω-stable theories	Stable theories	Simple theories

Indep. in AECs		
$\aleph_0$-stable homog.	Excellent classes	Finitary AECs

(In)dependence Logic

Dependence logic has established a basic theory of (in)dependence which encompasses the way (in)dependence behave in several disciplinary fields.

Two emblematic examples are **database theory** and **statistics**, where functional dependence and stochastic independence were shown to be completely symmetric to the way dependence and independence are treated in team semantics.

Convergence?

As we saw, in model theory a rich theory of independence has been elaborated. It comes then natural to ask: do the model-theoretic variants of the ubiquitous notion of (in)dependence fit into the framework of dependence logic?

In this talk we give a positive answer to this question, opening the way for new disciplinary interactions.

(In)dependence Atoms

We consider the following dependence and independence atoms:

$$=\!(\vec{x},\vec{y}), \quad \perp(\vec{x}), \quad \vec{x} \perp \vec{y}, \quad \perp_{\vec{z}}(\vec{x}) \quad \text{and} \quad \vec{x} \perp_{\vec{z}} \vec{y}.$$

Each kind of atom gives rise to an atomic language, and for each atomic language we define a team semantics and a deductive system.

Abstract Systems

The resulting systems are the following:

- atomic dependence logic;
- atomic absolute independence logic;
- atomic independence logic;
- atomic absolute conditional independence logic;
- atomic conditional independence logic.

In this talk we focus only on the first and the third system.

Atomic Dependence Logic – Semantics

The language of this logic is made only of dependence atoms. That is, let $\vec{x}$ and $\vec{y}$ be finite sequences of variables, then the formula $=(\vec{x}, \vec{y})$ is a formula of the language of atomic dependence logic.

Definition

Let $\mathcal{M}$ be a FO structure. Let $X = \{s_i\}_{i \in I}$ with $s_i : \mathrm{dom}(X) \to M$ and $\vec{x}\,\vec{y} \subseteq \mathrm{dom}(X) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $=(\vec{x}, \vec{y})$ under X, in symbols $\mathcal{M} \models_X =(\vec{x}, \vec{y})$, if

$$\forall s, s' \in X \; (s(\vec{x}) = s'(\vec{x}) \to s(\vec{y}) = s'(\vec{y})).$$

Atomic Dependence Logic – Deductive System

The following rules are known as the **dependence axioms**:

(a$_1$.) $=(\vec{x}, \vec{x})$ [as a degenerate case of this rule we admit $=(\emptyset, \emptyset)$];

(b$_1$.) If $=(\vec{x}, \vec{y})$, $\vec{u} \subseteq \vec{y}$ and $\vec{x} \subseteq \vec{z}$, then $=(\vec{z}, \vec{u})$;

(c$_1$.) If $=(\vec{x}, \vec{y})$ and $=(\vec{y}, \vec{z})$, then $=(\vec{x}, \vec{z})$;

(d$_1$.) If $=(\vec{x}, \vec{y})$ and $=(\vec{z}, \vec{v})$, then $=(\vec{x}\,\vec{z}, \vec{y}\,\vec{v})$;

(e$_1$.) If $=(\vec{x}, \vec{y})$, then $=(\pi\vec{x}, \pi\vec{y})$ [where $\pi : \mathrm{Var}^{<\omega} \to \mathrm{Var}^{<\omega}$ is the function that eliminates repetitions in finite sequences of variables];

(f$_1$.) If $=(\vec{x}, \vec{y})$, then $=(\vec{x}', \vec{y}')$ [where $\vec{x}' \in R[\vec{x}]$, $\vec{y}' \in R[\vec{y}]$ and $R : \mathrm{Var}^{<\omega} \to \mathrm{Var}^{<\omega}$ is the relation that add repetitions to finite sequences of variables].

Completeness

The system atomic dependence logic is sound and complete.

Theorem

Let Σ be a set of atoms, then

$$\Sigma \models =(\vec{x}, \vec{y}) \text{ if and only if } \Sigma \vdash =(\vec{x}, \vec{y}).$$

Proof.

See [Armstrong, 1974] and [Grädel and Väänänen, 2012]. ∎

Atomic Independence Logic – Semantics

The language of this logic is made only of independence atoms. That is, let $\vec{x}$ and $\vec{y}$ be finite sequences of variables, then the formula $\vec{x} \perp \vec{y}$ is a formula of the language of atomic independence logic.

Definition

Let $\mathcal{M}$ be a FO structure. Let $X = \{s_i\}_{i \in I}$ with $s_i : \mathrm{dom}(X) \to M$ and $\vec{x}\,\vec{y} \subseteq \mathrm{dom}(X) \subseteq \mathrm{Var}$. We say that $\mathcal{M}$ satisfies $\vec{x} \perp \vec{y}$ under X, in symbols $\mathcal{M} \models_X \vec{x} \perp \vec{y}$, if

$$\forall s, s' \in X\ \exists s'' \in X\ (s''(\vec{x}) = s(\vec{x}) \wedge s''(\vec{y}) = s'(\vec{y})).$$

Atomic Independence Logic – Deductive System

The following rules are known as the **independence axioms**:

(a_3.) $\vec{x} \perp \emptyset$;

(b_3.) If $\vec{x} \perp \vec{y}$, then $\vec{y} \perp \vec{x}$;

(c_3.) If $\vec{x} \perp \vec{y}\,\vec{z}$, then $\vec{x} \perp \vec{y}$;

(d_3.) If $\vec{x} \perp \vec{y}$ and $\vec{x}\,\vec{y} \perp \vec{z}$, then $\vec{x} \perp \vec{y}\,\vec{z}$;

(e_3.) If $x \perp x$, then $x \perp \vec{y}$ [for arbitrary $\vec{y}$];

(f_3.) If $\vec{x} \perp \vec{y}$, then $\pi\vec{x} \perp \sigma\vec{y}$ [where π and σ are permutations of $\vec{x}$ and $\vec{y}$ respectively].

Completeness

The system atomic independence logic is sound and complete.

Theorem

Let Σ be a set of atoms, then

$$\Sigma \models \vec{x} \perp \vec{y} \text{ if and only if } \Sigma \vdash \vec{x} \perp \vec{y}.$$

Proof.

See [Geiger and Paz and Pearl, 1991] and [Grädel and Väänänen, 2012]. ∎

Model-theoretic Interpretations

What happens if we interpret the (in)dependence atom in classes of structures in which a robust notion of (in)dependence has been defined? Is this interpretation sound with respect to the team-theoretic deductive system? Is it complete?

For example, what happens if we interpret the (in)dependence atom in a vector space and say that the atom is satisfied if the vectors are (in)dependent? What about algebraically closed fields and algebraic (in)dependence?

Things go smoothly in both cases, in fact we can prove a soundness and completeness result at a much greater level of generality.

Closure Operators

Definition

Let M be a set and $\mathrm{cl} : \mathcal{P}(M) \to \mathcal{P}(M)$ an operator on the power set of M. We say that cl is a closure operator and that (M, cl) is a closure system if for every $A, B \subseteq M$ the following conditions are satisfied:

i) $A \subseteq \mathrm{cl}(A)$;

ii) *If* $A \subseteq B$ *then* $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$;

iii) $\mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(A))$.

Examples

Example

Let (A, F) be an algebra of type Ω, and for every $B \subseteq A$ let $([B], F)$ be the subalgebra of (A, F) generated by B. Then $[\,] : \mathcal{P}(A) \to \mathcal{P}(A)$ is a closure operator.

Example

Let $\mathcal{M}$ be a first-order structure in the signature $\mathcal{L}$ and $A \subseteq M$. We say that b is algebraic over A if there is an $\mathcal{L}$-formula $\phi(v, \vec{w})$ and $\vec{a} \in A$ such that $\mathcal{M} \models \phi(b, \vec{a})$ and $\phi(\mathcal{M}, \vec{a}) = \{m \in M \mid \mathcal{M} \models \phi(m, \vec{a})\}$ is finite. Let $\mathrm{acl}_{\mathcal{M}}(A) = \{m \in M \mid m \text{ is algebraic over } A\}$. Then

$$\mathrm{acl}_{\mathcal{M}} : \mathcal{P}(M) \to \mathcal{P}(M)$$

is a closure operator.

Operators on Classes of Structures

Let $\mathcal{L}$ be a signature (possibly higher-order) and $\mathbf{K}$ be a class of $\mathcal{L}$-structures. We denote by $U(\mathbf{K})$ the class of domains of structures in $\mathbf{K}$ and by $\mathbf{V}$ the set theoretical universe. A second-order n-ary operator op on $\mathbf{K}$ is a class function $\mathrm{op} : U(\mathbf{K}) \to \mathbf{V}$ such that for every $\mathcal{M} \in \mathbf{K}$,

$$\mathrm{op}(M) : \mathcal{P}^n(M) \to \mathcal{P}^n(M).$$

Given such an operator, for every $\mathcal{M} \in \mathbf{K}$ we can consider the second-order structure $(M, \mathrm{op}(M))$. For ease of notation we denote the structure $(M, \mathrm{op}(M))$ simply as (M, op).

Closure Operator Atomic Dependence Logic

The syntax and deductive system of this logic are the same as those of atomic dependence logic.

Let $\mathcal{L}$ be a signature (possibly higher-order), $\mathbf{K}$ a class of $\mathcal{L}$-structures and cl a unary second-order closure operator on $\mathbf{K}$.

Definition

Let $\mathcal{M} \in \mathbf{K}$ and $s : \mathrm{dom}(s) \to M$ with $\vec{x}\,\vec{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. We let

$$\mathcal{M} \models_s =(\vec{x}, \vec{y}) \;\Leftrightarrow\; \forall y \in \vec{y}\; s(y) \in \mathrm{cl}(\{s(x) \mid x \in \vec{x}\}).$$

Completeness

Theorem

The system is sound.

Theorem

The system is complete if and only if there is $\mathcal{M} \in \mathbf{K}$ *such that* $\mathrm{cl}(\emptyset) \neq \emptyset$ *and* $\mathrm{cl}(\emptyset) \neq M$*.*

Abstract Elementary Classes

Definition (Abstract Elementary Class)

Let **K** *be a class of first-order structures in the signature* $\mathcal{L}$. *We say that* $(\mathbf{K}, \preccurlyeq)$ *is an* AEC *if the following conditions are satisfied.*

(1) **K** *and* $\preccurlyeq$ *are closed under isomorphism.*

(2) *If* $\mathcal{A} \preccurlyeq \mathcal{B}$, *then* $\mathcal{A} \subseteq \mathcal{B}$.

(3) *The relation* $\preccurlyeq$ *is a partial order on* **K**.

(4) *If* $(\mathcal{A}_i | \ < \delta)$ *is an increasing* $\preccurlyeq$*-chain, then:*

   (4.1) $\bigcup_{i<\delta} \mathcal{A}_i \in \mathbf{K}$;

   (4.2) *for each* $j < \delta$, $\mathcal{A}_j \preccurlyeq \bigcup_{i<\delta} \mathcal{A}_i$;

   (4.3) *if each* $\mathcal{A}_j \preccurlyeq \mathcal{M}$, *then* $\bigcup_{i<\delta} \mathcal{A}_i \preccurlyeq \mathcal{M}$.

(5) *If* $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}$, $\mathcal{A} \preccurlyeq \mathcal{C}$, $\mathcal{B} \preccurlyeq \mathcal{C}$ *and* $\mathcal{A} \subseteq \mathcal{B}$, *then* $\mathcal{A} \preccurlyeq \mathcal{B}$.

(6) *There is a Löwenheim-Skolem number* $\mathrm{LS}(\mathcal{K})$ *such that if* $\mathcal{A} \in \mathbf{K}$ *and* $B \subseteq A$, *then there is* $\mathcal{A}' \in \mathcal{K}$ *such that* $B \subseteq A'$, $\mathcal{A}' \preccurlyeq \mathcal{A}$ *and* $|A'| = |B| + |\mathcal{L}| + \mathrm{LS}(\mathcal{K})$.

Abstract Independence Relations in AECs

Definition (Abstract Independence Relation)

Let $(\mathbf{K}, \preccurlyeq)$ *be an* AEC *with* AP, JEP *and* ALM, *and* $\mathop{\smile\!\!\!\!\!\!|}$ *a ternary relation between (bounded) subsets of the monster model* $\mathfrak{M}$. *We say that* $\mathop{\smile\!\!\!\!\!\!|}$ *is a* pre-independence relation *if it satisfies the following axioms.*

(a.) *(Invariance) If* $A \mathop{\smile\!\!\!\!\!\!|}_C B$ *and* $f \in \mathrm{Aut}(\mathfrak{M})$, *then* $f(A) \mathop{\smile\!\!\!\!\!\!|}_{f(C)} f(B)$.

(b.) *(Existence)* $A \mathop{\smile\!\!\!\!\!\!|}_A B$, *for any* $A, B \subseteq \mathfrak{M}$.

(c.) *(Monotonicity) If* $A \mathop{\smile\!\!\!\!\!\!|}_C B$ *and* $D \subseteq A$, *then* $D \mathop{\smile\!\!\!\!\!\!|}_C B$.

(d.) *(Base Monotonicity) Let* $D \subseteq C \subseteq B$. *If* $A \mathop{\smile\!\!\!\!\!\!|}_D B$, *then* $A \mathop{\smile\!\!\!\!\!\!|}_C B$.

(e.) *(Symmetry) If* $A \mathop{\smile\!\!\!\!\!\!|}_C B$, *then* $B \mathop{\smile\!\!\!\!\!\!|}_C A$.

(f.) *(Transitivity) Let* $D \subseteq C \subseteq B$. *If* $B \mathop{\smile\!\!\!\!\!\!|}_C A$ *and* $C \mathop{\smile\!\!\!\!\!\!|}_D A$, *then* $B \mathop{\smile\!\!\!\!\!\!|}_D A$.

Abstract Independence Relations in AECs – Continued

Definition (Abstract Independence Relation – Continued)

(*g*.) *(Normality) If $A \downarrow_C B$, then $AC \downarrow_C B$.*

(*h*.) *(Finite Character) If $A_0 \downarrow_C B$ for all finite $A_0 \subseteq A$, then $A \downarrow_C B$.*

(*i*.) *(Anti-Reflexivity) If $A \downarrow_B A$, then $A \downarrow_B C$ for any $C \subseteq \mathfrak{M}$.*

If in addition $\downarrow$ satisfies the following two axioms, then we say that $\downarrow$ is an independence relation.

(*j*.) *(Extension) If $A \downarrow_C B$ and $B \subseteq D$, then there is $f \in \mathrm{Aut}(\mathfrak{M})$ fixing BC pointwise such that $f(A) \downarrow_C D$.*

(*k*.) *(Local Character) There is a cardinal $\kappa(\mathbf{K})$ such that for every $\vec{a} \in \mathfrak{M}^{<\omega}$ and $B \subseteq \mathfrak{M}$ there is $C \subseteq B$ with $|C| < \kappa(\mathbf{K})$ and $\vec{a} \downarrow_C B$.*

Elementary Examples

Example (Independence in *o*-minimal theories)

Let T be an o-minimal theory. For $A, B, C \subseteq \mathfrak{M}$, define $A \downarrow^{\mathrm{acl}}_{B} C$ if for every $\vec{a} \in A$ we have $\dim_{\mathrm{acl}}(\vec{a}/B \cup C) = \dim_{\mathrm{acl}}(\vec{a}/B)$. Then $\downarrow^{\mathrm{acl}}$ is a pre-independence relation.

Example (Forking in simple theories)

Let T be a simple theory. For $A, B, C \subseteq \mathfrak{M}$, define $A \downarrow^{\mathrm{frk}}_{B} C$ if for every $\vec{a} \in A$ we have that $\mathrm{tp}(\vec{a}/B \cup C)$ is a non-forking extension of $\mathrm{tp}(\vec{a}/B)$. Then $\downarrow^{\mathrm{frk}}$ is an independence relation.

Non-elementary Examples

Example (Pregeometric Independence)

Let $(\mathbf{K}, \preccurlyeq)$ *be an* AEC *with* AP, JEP *and* ALM, *and* $\mathrm{cl} : \mathfrak{M} \to \mathfrak{M}$ *a pregeometric operator. For* $A, B, C \subseteq \mathfrak{M}$, *define* $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_{B} C$ *if for every* $\vec{a} \in A$ *we have* $\dim_{\mathrm{cl}}(\vec{a}/B \cup C) = \dim_{\mathrm{cl}}(\vec{a}/B)$. *Then* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}$ *is a pre-independence relation.*

Example (Independence in Finitary AECs)

See [Hyttinen and Kesälä, 2006].

Independent Sequences

Definition (Independent Sequence)

Let $(\mathbf{K}, \preccurlyeq)$ *be* AEC *with* AP, JEP *and* ALM, *and* $\downarrow$ *a pre-independence relation. Let* $A \subseteq \mathfrak{M}$ *and* $(a_i \mid i \in I) \in \mathfrak{M}^I$ *injective. We say that* $(a_i \mid i \in I)$ *is an* independent sequence *over* A *if for all* $j \in I$ *we have that* $\{a_i \mid i < j\} \downarrow_A a_j$.

Definition (Constant tuple)

Let $(\mathbf{K}, \preccurlyeq)$ *be* AEC *with* AP, JEP *and* ALM, *and* $\downarrow$ *a pre-independence relation. We say that* $\vec{e} \in \mathfrak{M}^{<\omega}$ *is* constant *over* A *if* $\vec{e} \downarrow_A \vec{e}$. *We say that* $\vec{e} \in \mathfrak{M}$ *is constant if it is constant over* $\emptyset$.

Algebraic Independent Sequences

Definition (Algebraic Independent Sequence)

Let $\mathop{\smile\!\!\!\!\!\!|}$ be a pre-independence relation, $n \in \omega - \{0\}$ and $(a_i \mid i < n) \in \mathfrak{M}^n$ an independent sequence over A. We say that $(a_i \mid i < n)$ is algebraic over A if there exists $d \in \mathfrak{M}$ such that

$$d \not\mathop{\smile\!\!\!\!\!\!|}_A a_0 \cdots a_{n-1} \quad \text{and} \quad d \mathop{\smile\!\!\!\!\!\!|}_A \{a_i \mid i < n \text{ and } i \neq j\} \ \forall j < n.$$

We say that $(a_i \mid i < n)$ is algebraic if it is algebraic over $\emptyset$. We say that $\mathop{\smile\!\!\!\!\!\!|}$ is algebraic (over A) if it admits algebraic independent sequences (over A) of any finite length.

Abstract Independence Relation Atomic Independence Logic

The syntax and deductive system of this logic are the same as those of atomic independence logic.

Let $(\mathbf{K}, \preccurlyeq)$ be an AEC with AP, JEP and ALM, and $\downarrow$ a pre-independence relation between (bounded) subsets of the monster model $\mathfrak{M}$.

Definition

Let $s : \mathrm{dom}(s) \to \mathfrak{M}$ *with* $\vec{x}\,\vec{y} \subseteq \mathrm{dom}(s) \subseteq \mathrm{Var}$. *We let*

$$\mathfrak{M} \models_s \vec{x} \perp \vec{y} \iff s(\vec{x}) \underset{\emptyset}{\downarrow} s(\vec{y}).$$

Completeness

Theorem

The system is sound.

Theorem

The system is complete if and only if $\downfree$ *is algebraic and admits a constant point.*

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