Axiomatizing Modal Dependence Logic

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11th of March, 2014

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Outline

1. Modal logics
2. Axiomatisation through $\mathcal{ML}$
3. Team bisimulation
4. Coherence
5. Direct axiomatisation and decision methods for validity

Modal logics

The set of formulae for $\mathcal{ML}$ is generated by the following grammar

$$\varphi ::= p \mid \neg p \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Diamond\varphi \mid \Box\varphi.$$

The set of formulae for $\mathcal{ML}(\otimes)$ is generated by the following grammar

$$\varphi ::= p \mid \neg p \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \otimes \varphi) \mid \Diamond\varphi \mid \Box\varphi.$$

The set of formulae for $\mathcal{EMDL}$ is generated by the following grammar

$$\varphi ::= p \mid \neg p \mid \mathrm{dep}(\psi_1, \ldots, \psi_n) \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Diamond\varphi \mid \Box\varphi,$$

where $\psi_1, \ldots, \psi_n$ are formulae of $\mathcal{ML}$

Eliminating dependence atoms

Theorem (Väänänen 2008)

$\mathcal{MDL} \leq \mathcal{ML}(\varovee)$

Theorem (Ebbing, Hella, Meier, Müller, V., Vollmer 2013)

$\mathcal{EMDL} \equiv \mathcal{ML}(\varovee_{\mathcal{ML}})$. *(* $\mathcal{ML}(\varovee_{\mathcal{ML}})$ *is the syntactic fragment of* $\mathcal{ML}(\varovee)$ *in which the clause* $\varphi \varovee \varphi$ *is aplied only to* $\mathcal{ML}$*-formulae.)*

Disjunction property

Lemma (e.g, PhD thesis of Peter Lohmann, Sevenster 2009)

For every formula $\varphi \in \mathcal{ML}(\varovee)$ *there exists an equivalent formula*

$$\bigovee_{i \le n} \varphi_i,$$

where $\varphi_i \in \mathcal{ML}$ *for each* $i \le n$.

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Theorem (Sano and V.)

$\mathcal{ML}(\varovee)$ has the $\varovee$-disjunction property, i.e.,

$$\models \varphi \varovee \psi \quad \Rightarrow \quad \models \varphi \text{ or } \models \psi$$

Axiomatisation from $\mathcal{ML}$

Theorem (Sano and V.)

Axiomatisation for $\mathcal{ML}(\varovee)$ and complexity of the validity problem for $\mathcal{MDL}$, $\mathcal{EMDL}$ and $\mathcal{ML}(\varovee)$.

Proof.

Due to the disjuction property of $\mathcal{ML}(\varovee)$, we can construct an axiomatisation for $\mathcal{ML}(\varovee)$ by expanding some axiomatisation of $\mathcal{ML}$ with axioms that arise from the normal form theorem for $\mathcal{ML}(\varovee)$. Furthermore by translation from $\mathcal{EMDL}$ to $\mathcal{ML}(\varovee)$ we obtain a decision method for the validity problem for $\mathcal{MDL}$ and $\mathcal{EMDL}$. □

Team bisimulation

We say that K, T and K', T' are team bisimilar if and only if for each $w \in T$ there exists $w' \in T'$ such that K, w and K', w' are bisimilar, and vice versa.

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Lemma

If K, T and K', T' are team bisimilar, then K, T and K', T' agree on all formulae of $\mathcal{ML}(\ovee)$ and $\mathcal{EMDL}$.

Team k-bisimulation

We say that K, T and K', T' are team k-bisimilar if and only if for each $w \in T$ there exists $w' \in T'$ such that K, w and K', w' are k-bisimilar, and vice versa.

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Lemma

If K, T and K', T' are team k-bisimilar, then K, T and K', T' agree on all formulae of $\mathcal{ML}(\varovee)$ and $\mathcal{EMDL}$ of modal depth k.

Coherence

Definition

We say that $\varphi \in \mathcal{ML}(\varovee)$ (or $\mathcal{EMDL}$) is k-coherent if for every model K and team T of K

$$K, T \models \varphi \quad \Leftrightarrow \quad K, T' \models \varphi \text{ for every } T' \subseteq T \text{ s.t. } |T'| \leq k.$$

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Theorem (Sano and V.)

Every $\mathcal{ML}(\circledvee)$-formula φ is k-coherent for some $k \in \mathbb{N}$.

We say that a formula φ is hereditary k-coherent if every subformula of φ is k-coherent.

Coherence

Theorem (Sano and V.)

Every $\mathcal{ML}(\varovee)$-formula φ is (hereditary) k-coherent, for some $k \in \mathbb{N}$.

Coherence

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Every $\mathcal{ML}(\otimes)$-formula φ is (hereditary) k-coherent, for some $k \in \mathbb{N}$.

Theorem (Luosto)

Every $\mathcal{ML}(\otimes)$-formula φ is (hereditary) 2^k-coherent, where k is the number of $\otimes$ in φ.

Direct axiomatisation

Theorem (Sano and V.)

Axiomatisation and complexity of the validity problem for $\mathcal{MDL}$, $\mathcal{EMDL}$ and $\mathcal{ML}(\varovee)$.

Proof.

Due to the fact that every $\mathcal{ML}(\varovee)$ (and hence every $\mathcal{EMDL}$) formula is k-coherent, we can also construct directly a labelled calculus for each of the logics $\mathcal{MDL}$, $\mathcal{EMDL}$ and $\mathcal{ML}(\varovee)$. □

Labelled proofs

Idea:

Γ is a finite set of characteristic formulae (Hintikka formulae) of $\mathcal{ML}$ of some modal depth k and φ is a formula of $\mathcal{EMDL}$.

The intuition is that

$$\Gamma, \varphi \quad \text{iff} \quad \mathfrak{A}, T_\Gamma \models \varphi,$$

where $\mathfrak{A}$ is a canonical model for $\mathcal{ML}$ and T_Γ is a set of points in the canonical model that correspond to the set of Hintikka formulae Γ (i.e., for every $w \in T_\Gamma$ there exists some $\gamma \in \Gamma$ such that $\mathfrak{A}, w \models \gamma$, and vice versa).

Proof rules

Let $\Gamma(\tau, n)$ be the set of all Hintikka formulae of $\mathcal{ML}(\tau)$ of modal depth n. Let $\Gamma_1, \ldots, \Gamma_t$ be a sequence of all of the k size subsets of $\Gamma(\tau, n)$. If φ is a k-coherent $\mathcal{EMDL}$ formula of modal depth at most n, we have the following rule:

$$\frac{\Gamma_1, \varphi \qquad \Gamma_2, \varphi \qquad \Gamma_3, \varphi \qquad \ldots \qquad \Gamma_t, \varphi}{\varphi}$$

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$$\frac{\Gamma, \varphi \quad \Delta, \psi}{\Gamma \cup \Delta, \varphi \vee \psi}$$

$$\frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}$$

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$$\frac{\Gamma, \varphi \qquad \Delta, \psi}{\Gamma \cup \Delta, \varphi \vee \psi} \qquad \frac{\text{Each } \gamma \in \Gamma \text{ has } \neg p \text{ as a conjunct.}}{\Gamma, \neg p}$$

$$\frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi} \qquad \frac{\Gamma \text{ satisfies } \mathrm{dep}(\varphi_1, \ldots, \varphi_n)}{\Gamma, \mathrm{dep}(\varphi_1, \ldots, \varphi_n)}$$

Rules for modalities

Assume that $\Gamma_\Diamond$ is a finite set of Hintikka formulae of some modal depth k such that each $\psi \in \Gamma_\Diamond$ has $\Diamond\gamma$ as a conjuct for some $\gamma \in \Gamma$. Then we have the rule:

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Assume that $\Gamma_\Box$ is a finite set of Hintikka formulae of some modal depth k such that for every $\psi \in \Gamma_\Box$ and every conjunct $\Diamond\gamma$ of ψ it holds that $\gamma \in \Gamma$. Then we have the rule:

$$\frac{\Gamma, \varphi}{\Gamma_\Box, \Box\varphi}$$

Decision methods for the validity problem

Input: $\varphi \in \mathcal{EMDL}$

Method 1: Translate φ into $\varphi^* \in \mathcal{ML}(\varovee)$. Transform φ^* into φ^+ in $\varovee$-normalform. Check each disjunct of φ^+ by some decision method for *ML*.

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Method 2: Semantic tree -like method.

Semantic tree -like method

Idea: If φ is a (hereditary) k-coherent not valid formula then there exists some model and a team of size k that falsifies φ. We write: $F : \{a_1, \ldots, a_k\}, \varphi$.

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Rules:

$$\frac{F : \{a_1, \ldots, a_n\}, p}{F : \{a_1\}, p \quad \ldots \quad F : \{a_n\}, p}$$

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$$\frac{F : \{a_1, \ldots, a_n\}, \mathrm{dep}(\varphi_1, \ldots, \varphi_n)}{\text{Branch in every way such that } \{a_1, \ldots, a_n\} \text{ can falsify } \mathrm{dep}(\varphi_1, \ldots, \varphi_n)}$$

(e.g, label $F : \{a_1\}, \varphi_1, F : \{a_2\}, \varphi_1, T : \{a_1\}, \varphi_2,$
$T : \{a_2\}, \varphi_2, \ldots, T : \{a_1\}, \varphi_n, F : \{a_2\}, \varphi_n$)

More rules

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$$\frac{F : \{a_1, \dots, a_n\}, \Box\varphi}{\begin{array}{ccc} a_{i_1} R b_1 & \dots & a_{j_1} R b_1 \\ \vdots & & \vdots \\ a_{i_k} R b_k & & a_{j_k} R b_k \\ F : \{b_1, \dots, b_k\}, \varphi & & F : \{b_1, \dots, b_k\}, \varphi \end{array}}$$

Labels $b_1, \dots b_k$ are fresh and distinct.

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Labels $b_1, \dots b_k$ are fresh and distinct.

$$\frac{\begin{array}{c} a_1 R b_1 \\ \vdots \\ a_n R b_n \\ F : \{a_1, \dots, a_n\}, \Diamond\varphi \end{array}}{F : \{b_1, \dots, b_n\}, \varphi}$$

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